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SCUOLA NORMALE SUPERIORE
Classe di Scienze

AN INTRODUCTION TO DYNAMICAL SYSTEMS

by

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Preface

The material treated in this book was brought together for an introductory course I taught at the Scuola Normale Superiore in Pisa from january to may 2001. It is intended to be an introduction to dynamical systems. There are so many excellent introductions to dynamical systems that one needs to justify the write–up of these notes. My goal here is not at all to be complete but just to introduce some of the main ideas motivating the study of dynamical systems and their obiquitous presence in contemporary mathematics as well as to lead the reader in understanding some of the main conjectures. These lectures contain many problems (some of which may challenge the reader) : they should be considered as an essential part of the text. The proof of many useful and important facts is left as an exercise.

WARNING ! **This is a preliminary version ! ! ! ! ! ! ! ! ! ! !**

Pisa, November 19, 2001.

Stefano Marmi

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Part I. Basic Concepts

Lecture 1. Dynamical systems and group actions. Orbits. Structural stability

The most general definition of *dynamical system* is the action of a group (or a semigroup) G on some space X, called the *phase space.*

Definition 1.1 *A group G* acts on a differentiable manifold X *if there exists a map $\varphi : G \times X \to X$ such that :*

(i) for all $g \in G$ the map $\varphi_g : X \to X$, $\varphi_g(x) = \varphi(g, x)$, where $x \in X$ is a diffeomorphism ; if G is a Lie group we will also require φ to be smooth.

(ii) If e denotes the neutral element of G, φ_e =identity ;

(iii) for all $g_1, g_2 \in G$, $\varphi_{g_1 g_2} = \varphi_{g_1} \circ \varphi_{g_2}$.

This includes a huge number of possibilities. Roughly speaking, the goal of the theory of dynamical systems is to understand *most* of the dynamics of *most* systems. As we will see later understanding what is the meaning of "most" in the previous sentence is already doing some very important progress. Usually one considers a topological or a measure–theoretical meaning but also other possibilities are conceivable.

In looking for the simplest cases we are led to consider actions of $\mathbb{Z}$ (discrete time systems) or $\mathbb{R}, \mathbb{C}$ (continuous time systems) and to ask for the lowest possible dimension of the ambient space together with the highest possible regularity of the action. In the first case one has a diffeomorphism f of a smooth manifold X whereas in the continuous time case one considers the flow associated to a vector field v, i.e. the one–parameter group $(\varphi_t)_{t\in\mathbb{R}} \subset \mathrm{Diff}(X)$ of solutions of $\dot{x} = v(x), x(0) = x_0 : x(t, x_0) = \varphi_t(x_0)$. We are assuming the vector field $v : X \to TX$ to be complete, which is automatically satisfied if X is compact.

An important example is given by the actions of $G = \mathbb{Z}^2$ on X : to the canonical basis $\{(1, 0), (0, 1)\}$ of $\mathbb{Z}^2$ one can associate two generators $f, g \in \mathcal{G} = \mathrm{Diff}(X)$ so $f \circ g = g \circ f$. In this case the study of the action is equivalent to the study of a pair of commuting diffeomorphisms.

The action of a group on a manifold determines an equivalence relation on

the manifold : two points $x_1, x_2 \in X$ are *equivalent* if there exists an element $g \in G$ such that $x_2 = \varphi_g(x_1)$. Two equivalent points of the manifold belong to the same *orbit* $Gx = \{\varphi_g(x) \mid g \in G\}$ and the orbits of the points of X are the equivalence classes. Note that one can regard each orbit also as the image of the map $\psi_x : G \to X$, $\psi_g(x) = \varphi_g(x)$. A *fixed point* of the action is an orbit consisting of a single point.

The stabilizer H_x of the action at a point $x \in X$ is the set of elements $h \in G$ which leave x fixed $\varphi_h(x) = x$. H_x is a closed subgroup of G and ψ_x induces a 1–1 immersion of G/H_x into M.

The action of G on X is *free* if for $g \in G$ and $x \in X$, $g \neq e$ implies $\varphi_g(x) \neq x$. Equivalently $H_x = \{e\}$ for all x. The action is *discontinuous* if each $x \in X$ has a neighborhood U such that $U \cap \varphi_g(U) = \emptyset$ for all $g \in G$, $g \neq e$. The action is *transitive* if given any two points $x, y \in X$ there is $g \in G$ such that $\varphi_g(x) = y$.

The *quotient space* X/G is a topological space w.r.t. the topology induced asking that the projection $p : X \to X/G$ is continuous and open (the open subsets of X/G are the projection of the open subsets of X).

Exercise 1.2 Show that if the action of G on X is free and discontinuous X/G is a differentiable manifold and the projection is a local diffeomorphism. In this case X is an *(unramified) covering space* of X/G with the *covering transformation group* G. If the action is not free but it is dicontinuous then X is a *ramified covering* of X/G and the ramifying points (in X) are nothing but the points fixed by the action. Show that discontinuous actions always have finite stabilizers at all points.

Exercise 1.3 Consider the action of $\mathbb{Z}^k$ on $\mathbb{R}^l$ by translations and find necessary and sufficient conditions for the quotient being a smooth manifold. By definition if $k = l$ the quotient is the k–dimensional torus.

The strongest possible notion of equivalence between different actions on the same manifold is obtained asking that they are indeed the same action apart from a global change of coordinates on the manifold.

Definition 1.4 *Two actions φ, φ' of the same group G on X are* conjugate *if there is a diffeomorphism $h : X \to X$ such that $\varphi_g h = h\varphi'_g$ for all $g \in G$. If all the actions sufficiently close to a given one (w.r.t. a sensible topology) are conjugate to it we say that the given action is* rigid.

1.1 Discrete Dynamical Systems. Conjugation, Symmetries.

When $G = \mathbb{Z}$ one has a *discrete dynamical system* and the space of all possible dynamical systems on X is just the group $\mathcal{G} = \mathrm{Diff}\,(X)$ of diffeomorphisms of X. One can also consider actions of the semigroup $G = \mathbb{N}$, i.e. a homomorphism of G in the semigroup $\mathcal{G} = \mathrm{End}\,(X)$ of endomorphisms of X (smooth maps from X to X) : in this case φ_g is smooth but not necessarily invertible.

Let f be a generator of the action (i.e. $f \in \mathcal{G}$ and $\varphi_n = f^n$ for all $n \in G$, where f^n denotes the composition of f with itself n times if $n > 0$, $f^0 = \mathrm{id}_X$ is the identity, $f^{-n} = (f^{-1})^n = f^{-1} \circ \ldots \circ f^{-1}$). *From now on we will omit the symbol* $\circ$ *for the composition of two transformations* (unless some confusion may be possible). The *orbit* $\mathcal{O}_f(x)$ of $x \in X$ relative to f is the subset $\{f^n(x) \mid n \in G\}$. The finite orbits are called *periodic orbits* or *cycles* and their points *periodic points*. The *period* m of a periodic point x is the smallest positive integer such that $f^m(x) = x$. If $m = 1$ x is a *fixed point*. Note that all periodic points of f are fixed points of some iterate of f. A subset Y of X is *invariant* if $f(Y) = Y$.

Let X be a compact manifold. If $0 \leq r \leq \infty$ the space $\mathcal{G}^r$ of $\mathcal{C}^r$ discrete dynamical systems on X with the $\mathcal{C}^r$ topology is a Baire space and if $r < \infty$ it is even a Banach manifold. Similarly, if the *parameter space* P is also a compact manifold then the space of $\mathcal{C}^r$ families of discrete dynamical systems $F : P \times X \to X$ is also a Baire space. A *generic property* is a property that is true of dynamical systems belonging to some Baire set of $\mathcal{G}$ (a Baire set is the intersection of a countable number of open dense sets). When we deal with parametrized families we can also consider a different notion of typical : a property is *Lebesgue typical* if it is satisfied for maps corresponding to a full Lebesgue measure in the parameter space $P = \mathbb{R}^k$.

Definition 1.5 *Let $f \in \mathcal{G}$. We say that g is* equivalent *or* conjugate *to f if there is a diffeomorphism h of X such that*

$$f \sim g \iff \exists h \in \mathcal{G} \,:\, g = h^{-1} f h \,.$$

Remark 1.6 Sometimes one may compare also dynamical systems defined on two different phase spaces. If $f : X \to X$, $g : Y \to Y$ and $h : X \to Y$ is such that $gh = hf$ then conjugacy corresponds to h being a diffeomorphism. When h is an

embedding we say that f is a *subsystem* of g. When h is a *submersion*, f is an *extension* of g and g is a *factor* of f.

The set of diffeomorphisms equivalent to f obviously forms an equivalence class, the *orbit* of f under the adjoint action of $\mathrm{Diff}\,(X)$:

$$[f] = \mathrm{Ad}_{\mathrm{Diff}\,(X)} f = \{g \in \mathcal{G}\,,\ \exists h \in \mathrm{Diff}\,(X)\ : g = \mathrm{Ad}_h f = h^{-1} f h\}\ .$$

In some cases one can achieve a *complete* classification of the conjugacy classes [PY] but this is in general a totally unrealistic goal, as the following example shows.

Example 1.7 Let $\mathbb{C}\{z\}$ denote the ring of convergent power series in one variable. The germs of analytic diffeomorphisms of $\mathbb{C}$ with a fixed point at 0 are the elements of the group $\{f \in z\mathbb{C}\{z\}\,, f'(0) \neq 0\}$. We say that two such germs f and g are conjugate if there is another germ h such that $fh = hg$. Note that if f and g are conjugate then they have the same derivative $\lambda = f'(0)$ at the fixed point 0. If $|\lambda| \neq 1$ then a classical result of Poincaré and Koenigs says that λ is the only conjugacy invariant, i.e. all germs with the same derivative at the fixed point are conjugate. When $|\lambda| = 1$ the situation changes dramatically. Even in this apparently very simple context (a local problem in the smallest possible dimension with the highest possible regularity of the dynamics) a complete classification of the conjugacy classes of germs is open and perhaps unreasonable as the following result of Yoccoz [Y1] shows :

Theorem 1.8 *There is a generic set of values of $\lambda \in \mathbb{S}^1$ for which there exists a set with the power of the continuum of different conjugacy classes of germs all with linear part λ and each of which class does not contain an entire function.*

The proof of the previous result goes much beyond the scope of this introduction. We refer to [Y1] for its complete proof.

Exercise 1.9 Prove the Poincaré–Koenigs Theorem. [Hint : note that it is sufficient to show that if $|\lambda| \neq 1$ then all f with $f'(0) = \lambda$ are conjugate to the linear map $R_\lambda(z) = \lambda z$. Then one can apply either the contraction principle to the functional equation $f(h(z)) = h(\lambda z)$ or to prove directly the convergence of the Taylor expansion of h obtained by recurrence matching powers on the two sides of the functional equation. The latter is the classical method exposed also in [Ma1].]

If we drop the requirement in Definition 1.5 that h is a diffeomorphism and we only ask h to be a homeomorphism then f and g are *topologically conjugate* : in Lecture 2 we will study various invariants by topological conjugacy. Usually differentiable conjugacy is too a fine classification : indeed in order to be differentiably conjugated two dynamical systems must have the same derivative at all corresponding cycles. The notion of stability associated to topological conjugacy is called *structural stability* and is due to Andronov and Pontrjagin [AP].

Definition 1.10 *The set* $SS(X)$ *of* structurally stable discrete dynamical systems on X *consists of all systems* f *in* $\mathcal{G}$ *such that there is an open neighborhood* $\mathcal{N}$ *of* f *such that if* $g \in \mathcal{N}$ *then* f *and* g *are topologically conjugate.*

Clearly structurally stable dynamical systems form an open subset of $\mathcal{G}$. Its complement sometimes is also called the bifurcation set.

Definition 1.11 *A diffeomorphism* g *is a* symmetry *of* $f \in \mathcal{G}$ *if* $g \in Cent(f)$, *i.e. if* $Ad_g f = f$.

Exercise 1.12 Assume $g \sim f$ (i.e. $f = h^{-1}gh$) and show that
(1) Cent (f) is conjugated to Cent (g), i.e. Cent $(f) = h^{-1}$Cent $(g)h$;
(2) if $f(x) = x$ then $g(h(x)) = h(x)$ and $f'(x) = \lambda$ is invariant under conjugation : $g'(h(x)) = \lambda$;
(3) $f^{\mathbb{Z}} = \{f^n\ ,\ n \in \mathbb{Z}\} \subset$ Cent (f).

1.2 Flows vs. discrete dynamical systems. Cross–sections and suspensions

The relationship between discrete time dynamical systems and continuous time dynamical systems is very rich. Usually in dynamical systems one prefers to consider the discrete time case not just because of its great natural appeal but also because "the same phenomena and problems of the qualitative theory of ordinary differential equations are present in their simplest form in the diffeomorphism problem. Having first found theorems in the diffeomorphism case, it is usually a secondary tas to translate the results back into the differential equations framework" (Smale, [Sm] p. 747). Moreover many differential equations have

cross–sections, i.e. codimension 1 submanifolds Y of the phase space X which are transversal to the flow and on which the original problem of dynamics reduces to the study of the iteration of a diffeomorphism of Y. This method was first used by Poincaré more than a century ago in his study of the three body problem of celestial mechanics.

More recently another important fact has put emphasis on the study of discrete time dynamical systems : the massive use of computers to study numerically the solutions of differential equations. Virtually all methods of numerical integration of differential equations are based on a discretization of time and replace an o.d.e. (or a p.d.e.) with the iteration of a discrete time dynamical system. One should be however aware that this discretization is far from being an innocuous artifact ! Indeed even in the simplest cases the dynamics obtained can be on the long run dramatically different from the original one. For example consider the most classical mechanical example of a *simple pendulum*

$$\ddot{x} = \sin x \ .$$

A first order finite difference scheme of time step μ transforms it into

$$x(t+\mu) - 2x(t) + x(t-\mu) = \mu^2 \sin x(t) \ .$$

Setting $\varepsilon = \mu^2$, $x = x(t)$, $x' = x(t+\mu)$, $y = x(t) - x(t-\mu)$ and $y' = x(t+\mu) - x(t)$ one obtains the map of $\mathbb{T}^1_x \times \mathbb{R}_y$

$$x' = x + y' \ , \quad y' = \varepsilon \sin x + y$$

which is the celebrated *standard map* of the theory of twist maps (see Lecture 9). Whereas all the trajectories of the simple pendulum are either periodic or connect the unstable equilibrium position with itself in infininite time (separatrices) the trajectories of the standard map, even for very small values of ε, are extremely complicated, many aperiodic orbits exist and it is even conjectured that for all values of ε (different from zero) there exists a positive measure set of chaotic trajectories (in a very well precise sense, see Lecture 11).

Most of the concepts we define for discrete time dynamical systems have a natural reformulation for flows. In general we will leave to the reader the duty of translating them into the flow framework.

One exception is the notion of equivalence of two flows where various possibilities arise since one can choose to consider equivalent two flows with the same orbits (up to homeomorphisms) preserving or not the time evolution along them.

Definition 1.13 *A flow φ'_t on X is a* time reparametrization *of another flow φ_t on X if there exists a function $\chi : \mathbb{R} \times X \to \mathbb{R}$ (called an* (untwisted) one–cocycle *over φ_t) such that :*
(i) $\varphi'_t(x) = \varphi_{\chi(t,x)}(x)$ for all $x \in X$ and for all $t \in \mathbb{R}$;
(ii) $\chi(t,x) \geq 0$ if $t \geq 0$ and either $\chi(t,x) > 0$ if $t > 0$ or $\chi(t,x) \equiv 0$ (then x is a fixed point).

From the group properties of the flow one sees that χ must verify the equations

$$\chi(t+s,x) = \chi(t,x) + \chi(s,\varphi_t(x)) \ ,$$
$$\chi(-t,x) = -\chi(t,\varphi_{-t}(x)) \ .$$

One has now to possibilities : one can consider equivalence classes of flows up to time reparametrizations (*orbit equivalence*) or not (*flow equivalence*). In the former case one says that the flow φ_t is equivalent to ψ_t if there exists a diffeomorphism $h \in \mathrm{Diff}(X)$ such that $h\varphi_t h^{-1}$ is a time reparametrization of ψ_t. In the latter case one requires $\psi_t = h\varphi_t h^{-1}$, i.e. the two actions of $\mathbb{R}$ are conjugate. We will study time–reparametrizations of linear flows on tori in some detail in Lecture 3.

Exercise 1.14 Show that the vector fields of $\mathbb{R}^2$ $v(x,y) = (x,y)$ and $w(x,y) = (x+y, -x+y)$ give rise to topologically conjugate flows whereas they are not conjugate to $u(x,y) = (-y,x)$. Show that the vector field $v(x) = x^2$ of $\mathbb{R}$ is not structurally stable. Is u structrurally stable ?

Let X be a smooth manifold and let $\mathcal{X}(M)$ be the set of $\mathcal{C}^1$ (or $\mathcal{C}^r$) vector fields on X equipped with the $\mathcal{C}^1$ (resp. $\mathcal{C}^r$) topology.

Definition 1.15 *A vector field $v \in \mathcal{X}(M)$ is* structurally stable *if there is an open neighborhood $\mathcal{N}$ of v in $\mathcal{X}(M)$ such that all $w \in \mathcal{N}$ are $\mathcal{C}^0$–orbit equivalent to v (i.e. the flow $\varphi_{t,v}$ is topologically conjugate to a time reparametrization of $\varphi_{t,w}$).*

Once again structurally stable vector fields form an open subset of X. Peixoto [Pe] showed that the set of structurally stable vector fields (resp. diffeomorphisms) on a compact orientable 2–dimensional manifold X (resp. of the circle $\mathbb{T}^1$) is open and dense in $\mathcal{X}^r(M)$ (resp. $\mathrm{Diff}^r(\mathbb{T}^1)$) for $1 \leq r \leq \infty$. Indeed they do coincide with *Morse–Smale* systems (see [Sm3]) : these are the natural generalizations of gradient flows for which the Morse inequalities still hold [Sm1]. Unfortunately in

dimension ≥ 3 (for flows, resp. ≥ 2 for diffeomorphisms) structural stability is not a dense property [Sm2, Ne1]. A nice introduction to structural stability is due to Arnold [Ar4, Chapter 3].

We conclude this short survey on the relationship between flows and diffeomorphisms making the notion of cross-section more precise and showing conversely how to associate to each diffeomorphism a flow.

Definition 1.16 *A compact codimension one submanifold Y of a compact manifold X is called a* cross–section *for a flow φ_t on X if Y intersects every orbit, has transversal intersection with the flow and whenever $x \in Y$, $\varphi_t(x) \in Y$ for some $t > 0$.*

A flow φ_t induces a diffeomorphism $f \in \mathrm{Diff}\,(Y)$ by setting $f(x) = \varphi_{t_0}(x)$ where t_0 is the first positive t such that $\varphi_t(x) \in Y$. Note that there cannot be any fixed points for the flow whenever a cross–section exists. But one can have periodic orbits and they will become cycles (or fixed points) of f.

Exercise 1.17 Show that the topological equivalence of flows with cross–sections is determined by the topological equivalence of the corresponding diffeomorphisms.

Let us complete our discussion of cross–sections of flows showing how to associate to each diffeomorphism a flow.

Definition 1.18 *If f is a diffeomorphism of X the* suspension *of f is the flow φ_t on the manifold X_0 of dimension* $\dim X + 1$ *defined as follows :*

(i) X_0 is obtained as the orbit space of $X \times \mathbb{R}$ under the free action of $\mathbb{Z}$ generated by the diffeomorphism $\psi(x, u) = (f(x), u + 1)$;

(ii) The flow φ_t on X_0 is induced by the flow $\psi_t(x, u) = (f^{[t]}(x), u + t)$ on $X \times \mathbb{R}$.

Exercise 1.19 Show that X_0 has a cross–section $Y_0 = p(X \times \{0\})$ where $p : X \times \mathbb{R} \to X_0$ is the projection on the quotient. Check that the diffeomorphism f_0 associated to the cross–section Y_0 is differentiably conjugate to f.

Lecture 2. Topological properties of dynamics. Topological entropy

In this lecture we will first introduce several properties of a dynamical system which are preserved by topological conjugacy. We will also briefly discuss two topological invariants of the dynamics : the Artin–Mazur ζ function and topological entropy.

2.1 Recurrence, transitivity and minimality

Let f be a diffeomorphism.

Definition 2.1 *A point $x \in X$ is called a* wandering point *when there is a neighborhood U of x such that $\cup_{|n|>0} f^n(U) \cap U = \emptyset$. If f is not a diffeomorphism we only consider $n > 0$. A point will be called* nonwandering *if it is not a wandering point.*

These nonwandering points are those with the mildest possible form of recurrence. They form a closed invariant set usually denoted as $\Omega(f)$.

Given any point x we define the *α and ω limit sets* of its orbit

Definition 2.2 *A point $y \in X$ is called an* ω–limit point *(correspondingly, an* α–limit point*) of $x \in X$ if there is a sequence $(n_j)_{j\in\mathbb{N}}$ going to $+\infty$ (correspondingly, to $-\infty$) such that $f^{n_j}(x) \to y$. The set of all ω–limit points of x is denoted $\omega_f(x)$ or simply $\omega(x)$ (correspondingly, $\alpha_f(x)$ or $\alpha(x)$).*

From the definition of $\alpha(x)$ and $\omega(x)$ it follows that

$$\omega(x) = \bigcap_{n>0} \left(\overline{\bigcup_{m\geq n} f^m(x)} \right) ,$$

$$\alpha(x) = \bigcap_{n<0} \left(\overline{\bigcup_{m\leq n} f^m(x)} \right) .$$

Recurrent points must verify a much stronger constraint than nonwandering points :

Definition 2.3 *A point $x \in X$ is* positively recurrent *if $x \in \omega(x)$,* negatively recurrent *if $x \in \alpha(x)$,* recurrent *if it is both positively and negatively recurrent.*

Exercise 2.4 Show that $\Omega(f)$, $\alpha(x)$ and $\omega(x)$ are closed and invariant and that $\Omega(f)$ contains the ω– and α–limit sets of all points. Assume X compact and show that they cannot be empty.

Exercise 2.5 Give the definition of nonwandering point, α– and ω–limit sets and recurrent points for flows. Of course the properties of the above exercise are true also for flows.

Exercise 2.6 Consider the flows associated to the vector fields of $\mathbb{R}^2$ $v(x,y) = (x,-y)$ and $w(x,y) = (y + x(1-x^2-y^2), -x + y(1-x^2-y^2))$ and determine the α– and ω–limit sets of all orbits. Show that $[0,+\infty) \times \{0\}$ is a cross–section of the flow associated to w and compute the induced diffeomorphism f. [Answer : $f(x) = e^{\pi}x/\sqrt{1-x^2+x^2e^{2\pi}}$.]

As we have seen in Lecture 1, it was already known in the sixties that the notion of structural stability was too strong and that structurally stable systems are not dense. Thus Smale and Palis proposed the notion of Ω–stability.

Definition 2.7 *Two diffeomorphisms f and g are* Ω–conjugate *if there is a homeomorphism $h : \Omega(f) \to \Omega(g)$ such that $hf = gh$. f is called* Ω–stable *if there is a neighborhood $\mathcal{N}$ of f in $Diff(X)$ such that all $g \in \mathcal{N}$ are Ω–conjugate to f.*

Smale and Palis used this concept in their formulation of the Stability Conjecture which we will discuss in Lecture 4.

Periodic points are the simplest kind of orbits one can conceive and they give the most perfect case of recurrence. However they do not always exists. The next case of very strong and uniform recurrence is represented by *minimal sets.*

Definition 2.8 *A discrete dynamical system $f : X \to X$ is* topologically transitive *if there exists a point $x \in X$ such that its orbit $\mathcal{O}_f(x)$ is dense in X. If the orbit of every point is dense in X then f is called* minimal. *A closed non–empty f–invariant subset A of X is called a* minimal set *if $f|_A$ is minimal.*

Exercise 2.9 Let X be compact. Show that the following are equivalent :

(i) f is minimal ;
(ii) X has no proper closed non–empty f–invariant subset ;
(iii) for every non–empty open set U of X one has $\cup_{-\infty}^{+\infty} f^n(U) = X$.

We remark that if A is a minimal set and γ is an orbit contained in A then γ is recurrent. This is because $\omega(\gamma)$ is closed, nonempty, invariant and $\omega(\gamma) \subset A$. Thus $\gamma \subset A = \omega(\gamma)$. Similarly $\gamma \subset \alpha(\gamma)$.

Minimality has an interesting consequence on the observables, i.e. on continuous functions $F : X \to \mathbb{C}$.

Proposition 2.10 *If $f : X \to X$ is a minimal homeomorphism of a compact manifold X then the only f–invariant continuous functions are the constants.*

Proof. Let $F \circ f = F$. Then the same holds iterating f n times $F \circ f^n = F \; \forall n \in \mathbb{Z}$. Pick any point $x \in X$. Then its orbit is dense in X and F must be constant on the orbit. By continuity F is constant on X. □

Of course a given discrete dynamical system need not be minimal. But it always has a minimal set.

Proposition 2.11 *Every continuous map f on a compact metric space X has a minimal set.*

Proof. Let $\mathcal{A}$ denote the collection of all closed f–invariant non–empty closed subsets of X. Inclusion gives a partial ordering on $\mathcal{A}$. Every ordered chain in $\mathcal{A}$ has a non–empty least element (the intersection of the elements of the chain, which is clearly closed, non–empty and invariant). Thus by Zorn's lemma $\mathcal{A}$ has a minimum element which is by construction a minimal set. □

Denjoy [De] constructed a $\mathcal{C}^1$ vector field of the two–dimensional torus $\mathbb{T}^2$ with a nontrivial minimal set (i.e. distinct from a fixed point or a periodic orbit) distinct from $\mathbb{T}^2$ (see also Appendix 1). On the other hand one can prove [Sch] that a minimal set A of a $\mathcal{C}^2$ vector field on a two–dimensional compact surface X is either trivial or the whole of X and, in this case, $X = \mathbb{T}^2$.

The *Cherry flow* [Ch] is an example of an analytic vector field on the two–torus with highly nontrivial recurrence. It has exactly two fixed points, no closed orbit and it has a circle cross–section and the non–wandering set of the induced

diffeomorphism is the union of an attracting fixed point and an invariant Cantor set (i.e. a compact, totally disconnected, nonempty perfect set). We refer to [PDM] (pp. 181–188) for its construction.

Exercise 2.12 Let G be a topological group and consider the action of G on itself by left translations : $L : G \times G \to G$, $L_g(g') = L(g, g') = gg'$. Show that if the discrete dynamical system on G generated by the left translation L_g by a fixed element g is topologically transitive then it is also minimal.

The previous exercise shows that in the case of translations on a topological group being topologically transitive is equivalent to being minimal. The example of translations on abelian groups is the prototype of quasiperiodic dynamics (see Lecture 3). We now see some criteria for topological transitivity (see [Wa], p. 127)

Proposition 2.13 *Let $f : X \to X$ be a homeomorphism of a compact metric space X. The following are equivalent :*

(i) f is topologically transitive ;

(ii) if U is open and $f(U) = U$ then either U is empty or U is dense ;

(iii) for any two non–empty open sets U, V there exists an integer $N = N(U, V)$ such that $f^N(U) \cap V \neq \emptyset$;

(iv) the set of points of X with a dense orbit is a dense G_δ.

Proof. (i)⇒(ii). Let the orbit of $x \in X$ be dense and let $U \neq \emptyset$, U open and $f(U) = U$. There exists an integer j such that $f^j(x) \in U$, thus by the f–invariance of U the orbit of x is contained in U and U is dense.
(ii)⇒(iii). Let U, V be two open non–empty sets. Then the invariant non–empty open set $\cup_{n\in\mathbb{Z}} f^n(U)$ is necessarily dense. Thus $\cup_{n\in\mathbb{Z}} f^n(U) \cap V \neq \emptyset$ from which (iii) follows.
(iii)⇒(iv). Let $(U_n)_{n\in\mathbb{Z}}$ be a countable open basis for X. Then $\{x \in X \mid \overline{\mathcal{O}_f(x)} = X\} = \cap_{n=1}^{\infty} \cup_{m\in\mathbb{Z}} f^m(U_n)$ and by (iii) $\cup_{m\in\mathbb{Z}} f^m(U_n)$ is dense.
(iv)⇒(i). Obvious. □

2.2 Artin–Mazur's ζ function

Let f be an endomorphism of compact manifold X. We denote $N_m = N_m(f)$ the cardinality of the set $\{x \in X \mid f^m(x) = x\,,\ x \text{ isolated}\}$, i.e. the number of isolated

periodic points of f with period p which divides m. The relation with the number K_m of periodic points with primitive period m is given by the Möbius inversion formula :

$$N_m = \sum_{l|m} K_l \ , \quad K_m = \sum_{l|m} \mu(l) N_{m/l} \ ,$$

where μ is the Möbius function :

$$\mu(l) = \begin{cases} 1 & \text{if } l = 1, \\ (-1)^r & \text{if } l \text{ is the product of } r \text{ distinct primes,} \\ 0 & \text{if there is a prime } p \text{ such that } p^2 \text{ divides } m. \end{cases}$$

One can prove that generically $N_m < \infty$ for all $m \in \mathbb{N}$. Artin and Mazur [AM] introduced the ζ function of a diffeomorphism as the formal power series $\zeta(t) = \exp\left(\sum_{m=1}^{\infty} N_m t^m / m\right)$. Note that m always divides K_m.

Of course the zeta function of f is invariant for topological conjugacy and also for Ω–conjugacy. The main result of Artin and Mazur is the following

Theorem 2.14 *Let X be a compact manifold and $1 \leq r < \infty$. There is a dense set $\mathcal{D}^r$ in $End^r(X) = \mathcal{C}^r(X, X)$ (with the uniform $\mathcal{C}^r$ topology) such that if $f \in \mathcal{D}^r$ then there exist two positive constants C, k such that*

$$N_m \leq C k^m \ , \forall m \in \mathbb{N} \ .$$

Thus for a dense set of discrete dynamical systems on a compact manifold the zeta function has a positive radius of convergence.

Artin and Mazur used a famous result of John Nash to prove Theorem 2.14. In [N1] Nash proved that every smooth compact manifold can be realized as a sheet of a real algebraic variety. Only very recently Kaloshin [Kal1] has given a much more elementary argument based on Weierstrass approximation theorem : his proof will be discussed in Appendix 2. He also proved [Kal2] the following surprising and remarkable result

Theorem 2.14' *Let X be a compact manifold and $2 \leq r < \infty$. The set of diffeomorphisms $f \in Diff^r(X)$ with at most exponential growth of $N_m(f)$ is* not generic *in $Diff^r(X)$ (with the uniform $\mathcal{C}^r$ topology).*

Kaloshin's theorem gives a negative answer to a question of Smale [Sm, Problem 4.5, p. 765] who had asked if the Artin–Mazur ζ functions were generically rational

functions. Moreover Theorem 2.14' answers negatively to a question of Bowen relating topological entropy with the asymptotic growth of the number of periodic orbits (see Remark 2.26).

The simplest way to measure the asymptotic growth of the number of periodic points is the exponential growth rate $r(f)$ of the sequence $N_m(f)$

$$r(f) = \limsup_{m\to\infty} \frac{1}{m} \log \max(N_m(f), 1) \, .$$

Of course ζ converges in the complex disk $|t| < e^{-r(f)}$ and always has a singularity on its circle of convergence. Often ζ admits an analytic continuation to a meromorphic function of the whole complex plane. This is for example the case of Axiom A diffeomorphisms for which the zeta function is rational [Man].

Exercise 2.15 Show that the ζ function for the p–adic map on $[0,1]$, $x \mapsto px (\mathrm{mod}\, 1)$, is $\zeta(t) = (1-t)/(1-pt)$.

Exercise 2.16 Show that

(i) $\zeta(t) = \prod_{m=1}^{\infty} \prod_{\mathcal{O}_f(x) \text{has period } m} \frac{1}{1-t^m}$;

(ii) If all the periodic points of f are contained in the union of two disjoint closed invariant sets Λ_1 and Λ_2 then the ζ function of f is the product of the ζ functions of $f|_{\Lambda_1}$ and $f|_{\Lambda_2}$.

2.3 Topological Entropy

The most important numerical invariant related to the orbit growth in dynamical systems is *topological entropy*. It represents the exponential growth rate of the number of orbit segments which are distinguishable with an arbitrarily high but finite precision. Again, topological entropy is invariant by topological conjugacy (Proposition 2.24). We will follow the definition given by Rufus Bowen in [Bo, Chapter 4]. Let X be a compact metric space.

Definition 2.17 *Let $S \subset X$, $n \in N$ and $\varepsilon > 0$. S is a* (n,ε)–spanning set *if for every $x \in X$ there exists $y \in S$ such that $d(f^j(x), f^j(y)) \le \varepsilon$ for all $0 \le j \le n$.*

It is immediate to check that the compactness of X implies the existence of finite spanning sets. Let $r(n,\varepsilon)$ be the least number of points in an (n,ε)–spanning set. If we bound the time of observation of our dynamical system by n and our precision in making measurements is ε we will see at most $r(n,\varepsilon)$ orbits.

Exercise 2.18 Show that if X admits a cover by m sets of diameter $\leq \varepsilon$ then $r(n, \varepsilon) \leq m^n$.

Definition 2.19 *The* topological entropy $h_{top}(f)$ *of* f *is given by*

$$h_{top}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log r(n, \varepsilon) \ .$$

Note that Exercise 2.18 shows that the topological entropy for any diffeomorphism of a compact manifold is always finite.

In the previous definition one cannot replace lim sup with lim since there exist examples of maps for which the limit does not exist. However one can replace it with lim inf still obtaining the topological entropy (see [Mn1], Proposition 7.1, p. 237).

Exercise 2.20 Show that the topological entropy of the p–adic map of Exercise 2.15 is $\log p$. Compare with the growth rate $r(f)$ of the number of periodic orbits.

Remark 2.21 In the noncompact case one can define topological entropy as follows : take any compact subset K of X and consider (n, ε)–spanning sets of K (i.e. $x \in K$ in Definition 2.17). Denote $r(n, \varepsilon, K)$ the smallest cardinality of any (n, ε)–spanning set of K. Then set

$$h_{top}(f) = \sup_{K} \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log r(n, \varepsilon, K) \ .$$

Remark 2.22 The topological entropy of a flow φ_t is defined as the topological entropy of the time–one diffeomorphism $f = \varphi_1$.

Exercise 2.23 Show that :

(i) the topological entropy of an isometry is zero ; if h is an isometry the topological entropy of f equals that of $h^{-1} \circ f \circ h$.

(ii) if f is a homeomorphism of a compact space X then $h_{top}(f) = h_{top}(f^{-1})$;

(iii) $h_{top}(f^m) = |m| h_{top}(f)$.

Exercise 2.24 Let X be a metric space and f a continuous endomorphism of X. We say that a set A is (n, ε)–separated if for all $x, y \in X$ there exists a $0 \leq j \leq n$ such that $d(f^j(x), f^j(y)) > \varepsilon$. We denote $s(n, \varepsilon)$ the maximal cardinality of an (n, ε)–separated set. Show that :

(i) $s(n,2\varepsilon) \le r(n,\varepsilon) \le s(n,\varepsilon)$;
(ii) $h_{top}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log s(n\varepsilon)$;
(iii) if X is a compact subset of $\mathbb{R}^l$ and f is Lipschitz with Lipschitz constant K then $h_{top}(f) \le l \log K$.

Proposition 2.25 *The topological entropy does not depend on the choice of the metric on X provided that the induced topology is the same. The topological entropy is invariant by topological conjugacy.*

Proof. We first show how the second statement is a consequence of the first. Let f, g be topologically conjugate via a homeomorphism X. Let d denote a fixed metric on X and d' denote the pullback of d via h : $d'(x_1,x_2) = d(h^{-1}(x_1), h^{-1}(x_2))$. Then h becomes an isometry so $h_{top}(f) = h_{top}(g)$ (see Exercise 2.22).

Let us now show the first part. Let d and d' be two different metrics on X which induce the same topology and let $r_d(n,\varepsilon)$ and $r_{d'}(n,\varepsilon)$ denote the minimal cardinality of a (n,ε)-spanning set in the two metrics. We will denote $h_{top,d}(f)$ and $h_{top,d'}(f)$ the corresponding topological entropies.

Let $\varepsilon > 0$ and consider the set D_ε of all pairs $(x_1,x_2) \in X \times X$ such that $d(x_1,x_2) \ge \varepsilon$. This is a compact subset of $X \times X$ thus d' takes a minimum $\delta'(\varepsilon) > 0$ on D_ε. Thus any $\delta'(\varepsilon)$–ball in the metric d' is contained in a ε–ball in the metric d. From this one gets $r_{d'}(n,\delta'(\varepsilon)) \ge r_d(n,\varepsilon)$ thus $h_{top,d'}(f) \ge h_{top,d}(f)$. Interchanging the role of the two metrics one obtains the opposite inequality. □

Remark 2.26 Bowen [Bo, p. 23] asked the following question : does $h_{top}(f) = r(f) = \limsup_{m\to\infty} \frac{1}{m} \log N_m(f)$ for a $\mathcal{C}^r$ $(r \ge 1)$ generic set of diffeomorphisms $f \in \mathrm{Diff}^r(X)$? Kaloshin's Theorem 2.14' gives a negative answer if $r \ge 2$.

Exercise 2.27 Show that if g is a factor of f then $h_{top}(g) \le h_{top}(f)$.

An alternative but equivalent definition of topological entropy is obtained considering all possible open covers of X and their refinements obtained by iterating f.

Definition 2.28 *If α, β are open covers of X theire* join *$\alpha \vee \beta$ is the open cover by all sets of the form $A \cap B$, where $A \in \alpha$ and $B \in \beta$. An open cover β is a* refinement *of α, written $\alpha < \beta$, if every member of β is a subset of a member of α.*

Let α be an open cover of X and let $N(\alpha)$ be the number of sets in a finite subcover of α with smallest cardinality. We denote $f^{-1}\alpha$ the open cover consisting of all sets $f^{-1}(A)$ where $A \in \alpha$.

Exercise 2.29 If $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers such that $a_{n+m} \leq a_n + a_m$ for all n, m then $\lim_{n\to+\infty} a_n/n$ exists and equals $\inf_{n\in\mathbb{N}} a_n/n$.

Theorem 2.30 *The topological entropy of f is given by*

$$h_{top}(f) = \sup_{\alpha} \lim_{n\to\infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right) .$$

Lecture 3. Quasiperodic orbits

A periodic orbit of a dynamical system is a closed orbit. Among all recurrent orbits, almost periodic orbits are those which behave as periodic orbits if one looks at the phase space with a finite resolution. If the resolution is increased the orbit seems again periodic but with a longer period. *Quasiperiodic* orbits have the further property that the frequencies of the motion (which one can obtain by Fourier analysis) span a finite dimensional space.

A good way to introduce formally this concepts is to follow the approach of signal theory.

3.1 Almost periodic functions

Definition 3.1 *Let $F : \mathbb{R} \to \mathbb{C}$ be a continuous function and let $\varepsilon > 0$. An* ε–almost period *of F is a number $\tau \in \mathbb{R}$ such that*

$$\sup_{t\in\mathbb{R}} |F(t+\tau) - F(t)| < \varepsilon \ .$$

A function F as above is almost periodic *if for all $\varepsilon > 0$ there exists $L(\varepsilon, F) > 0$ such that any interval of length L on $\mathbb{R}$ contains an ε–almost period.*

Exercise 3.2 Show that
(i) if F is almost periodic so are $|F|$, $\bar{F}$, aF, $F(at)$;
(ii) almost periodic functions are bounded ;
(iii) if F is almost periodic so is F^2 ;
(iv) almost periodic functions are uniformly continuous.

Exercise 3.3 Let $\mathcal{P}(\varepsilon, F)$ denote the set of all ε–almost periods of F. Show that
(i) $\mathcal{P}(\varepsilon_1, F) + \mathcal{P}(\varepsilon_2, F) \subset \mathcal{P}(\varepsilon_1 + \varepsilon_2, F)$;
(ii) $\mathcal{P}(\varepsilon_1, F) \cap \mathcal{P}(\varepsilon_2, G) \subset \mathcal{P}(\varepsilon_1 + \varepsilon_2, F + G)$;
(iii) for all $\varepsilon > 0$ there exists $\delta > 0$ such that $[-\delta, \delta] \subset \mathcal{P}(\varepsilon, F)$;
(iv) for all $\varepsilon_1, \varepsilon_2 > 0$ there exists $\delta > 0$ such that $\mathcal{P}(\varepsilon_1, F) + [-\delta, \delta] \subset \mathcal{P}(\varepsilon_1 + \varepsilon_2, F)$.

Exercise 3.4 Using the results of the previous exercises show that the sum of two almost periodic functions is almost periodic. Thus any *trigonometric polynomial* $F(t) = A_1 e^{2\pi i \alpha_1 t} + \ldots + A_k e^{2\pi i \alpha_k t}$ is almost periodic. The numbers α_i are called the *frequencies* of F, the numbers $A_i \in \mathbb{C}$ are called the *amplitudes*.

Theorem 3.5 *A bounded continuous function F is almost periodic if and only if the family of its translates $\{F(\cdot+s)\}_{s\in\mathbb{R}}$ is sequentially precompact (in the uniform norm)*

Proof. Let us first recall that $\{F(\cdot+s)\}_{s\in\mathbb{R}}$ is sequentially precompact if and only if it is totally bounded, i.e. for every $\varepsilon>0$ it can be covered by a union of a finite number of balls of radius ε w.r.t. the uniform norm $\|F\|=\sup_{t\in\mathbb{R}}|F(t)|$.

Suppose that F is almost periodic. Then an almost period of F is clearly an almost period of $F(\cdot+s)$. Let $\varepsilon>0$ and let $L=L(\varepsilon/2,F)$; by the uniform continuity of F (Exercise 3.2 (iv)) we can find $t_1,\dots,t_M\in[0,L]$ such that if $t_0\in[0,L]$ then $\inf_{1\le j\le M}\sup_{s\in\mathbb{R}}|F(s+t_0)-F(s+t_j)|\le\varepsilon/2$. Given $t\in\mathbb{R}$ let $\tau\in\mathcal{P}(\varepsilon/2,F)\cap[t-L,t]$. Then setting $t_0=t-\tau$ we obtain $\sup_{s\in\mathbb{R}}|F(s+t_0)-F(s+t)|\le\varepsilon/2$ thus $\{F(\cdot+s)\}_{s\in\mathbb{R}}$ is covered by the union of the balls of radius ε centered at $F(\cdot+t_j)$, $j=1,\dots M$.

Conversely assume F is not almost periodic. Then there is some $\varepsilon>0$ and a sequence I_j of intervals whose lengths $|I_j|\to+\infty$ as $j\to\infty$ such that I_j contains no ε–almost period of f. We will now construct a sequence s_j such that $F(\cdot+s_j)$ has no convergent subsequence. Let $s_1=0$, $s_2=s_2-s_1\in I_1$. Choose j_1 such that $|I_{j_1}|>|s_2-s_1|$ and s_3 such that $s_3-s_2\in I_{j_1}$. Proceeding in this way we can arrange that $s_i-s_k\in I_{j_i}$ for all $k\le i$. Thus for any i and k we conclude that s_i-s_k is not an ε–almost period of F. Thus

$$\sup_{s\in\mathbb{R}}|F(s+s_i)-F(s+s_k)|=\sup_{t\in\mathbb{R}}|F(t+s_i-s_j)-F(t)|>\varepsilon$$

and no subsequence of $\{F(\cdot+s_j)\}_{j\in\mathbb{N}}$ is Cauchy. □

Exercise 3.6 Show that if G is the uniform limit of a sequence of almost periodic functions then it is almost periodic.

The previous Theorem has the following important Corollary :

Corollary 3.7 *Let $\mathcal{F}$ be a uniformly continuous function of k complex variables. If $F_1,\dots,F_k$ are almost periodic functions then $\mathcal{F}(F_1(\cdot),\dots,F_k(\cdot))$ is almost periodic.*

Proof. It is enough to note that if each of the families of translates $\{F_i(\cdot+s)\}_{s\in\mathbb{R}}$ is sequentially precompact then so is $\{\mathcal{F}(F_1(\cdot+s),\dots,F_k(\cdot+s))\}_{s\in\mathbb{R}}$. □

3.2 Quasiperiodic functions and linear flows on the torus

Among almost periodic functions the most relevant case for dynamics is provided by quasiperiodic functions.

Definition 3.8 *A function F is* quasiperiodic *if there exists a uniformly continuous function $\mathcal{F}$ of n complex variables and n real numbers $\alpha_1, \dots, \alpha_k$ such that*

$$F(t) = \mathcal{F}(e^{2\pi i \alpha_1 t}, \dots, e^{2\pi i \alpha_n t}) \, . \tag{3.1}$$

Remark 3.9 Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denote the n–dimensional torus. Condition (3.1) is equivalent to

$$F(t) = \Phi \circ \varphi_{\alpha,t}(x_0) \, , \tag{3.2}$$

where x_0 is some point on $\mathbb{T}^n$, $\alpha in \mathbb{R}^n$ $\Phi : \mathbb{T}^n \to \mathbb{C}$ is uniformly continuous and $\varphi_{\alpha,t}$ is the *linear flow on $\mathbb{T}^n$ with speed α*, i.e. the time t solution of $\dot{x} = \alpha$, $x(0) = x_0$.

Remark 3.10 One can prove that every almost periodic function can be approximated uniformly by quasi periodic functions (Bohr's Theorem). [Hint : see [St], Vol. I, or [Kat], p. 165]

Let $\mathcal{M} = \{k \in \mathbb{Z}^n \mid k \cdot \alpha = 0\}$. Note that $\mathcal{M}$ is a $\mathbb{Z}$–module. If $\dim \mathcal{M} = n$ then $\alpha = 0$, if $\dim \mathcal{M} = n - 1$ then there exists $\beta \in \mathbb{R}$ and $k \in \mathbb{Z}^n$ such that $\alpha = \beta k$ and all orbits of $\varphi_{\alpha,t}$ are periodic. If $\dim \mathcal{M} = 0$ then α is called *non–resonant*.

Exercise 3.11 Show that the closure of any orbit of the linear flow $\dot{x} = \alpha$ on $\mathbb{T}^n$ is diffeomorphic to the torus $\mathbb{T}^{n-\dim \mathcal{M}}$.

Exercise 3.12 Show that if $\dim \mathcal{M} \in \{1, \dots n-1\}$ there exists $A \in \mathrm{SL}(n, \mathbb{Z})$ such that posing $y = Ax$ the linear flow $\dot{x} = \alpha$ on $\mathbb{T}^n$ becomes $\dot{y}_i = 0$ for $i = 1, \dots, m$ and $\dot{x}_i = \alpha'_i$ for $i = m+1, \dots, n$ with $(\alpha'_{m+1}, \dots, \alpha'_n) \in \mathbb{R}^{n-m}$ non–resonant.

Exercise 3.13 Show that if α is non–resonant then the Haar measure on $\mathbb{T}^n$ is uniquely ergodic (see [Mn] for its definition) for the linear flow on $\mathbb{T}^n$.

The prototype of a quasiperiodic dynamical system is given by the (non–resonant) linear flow on the torus in the continuous case and by a translation on the n–dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ in the discrete time case. If α and x are now

two points of $\mathbb{T}^n$ we define $R_\alpha x = x + \alpha (\mathrm{mod}\, \mathbb{Z}^n)$. One sees immediately three important features of this example :

Exercise 3.14 Show that the translations on the torus have the following properties :

- from the *algebraic* point of view, the centralizer of R_α is the whole torus $\mathbb{T}^n$. The dynamics is homogenous and the group of symmetries acts transitively on the phase space ;
- from the *topological* point of view, the family of iterates $(R_\alpha^n)_{n\in\mathbb{Z}}$ is equicontinuous. The topological entropy of R_α is zero ;
- from the *measure–theoretical* point of view, the Haar measure on $\mathbb{T}^n$ is invariant under R_α and the unitary operator U_{R_α} on $L^2(\mathbb{T}^n, \mathbb{C})$ defined by $U_{R_\alpha} F = F \circ R_\alpha$ has discrete spectrum $\{e^{2\pi i k\cdot\alpha}\}_{k\in\mathbb{Z}^n}$.

3.3 Time–reparametrizations of linear flows : diophantine conditions

The translation flow on the torus $\mathbb{T}^n$ of vector $\alpha \in \mathbb{R}^n$ is the flow arising from the constant vector field $X(x) = \alpha$. We denote this flow by $R_{t\alpha}$. When the vector α is non resonant, i.e. when $\alpha_1, \ldots, \alpha_n$ are rationally independent, the flow is minimal and has a unique invariant probability measure which is the Haar measure on $\mathbb{T}^n$. In this case we say it is an *irrational flow*. Note that one of the coordinates of the corresponding vector field might be rational. More specifically, given a minimal translation R_α on $\mathbb{T}^n$ then the flow $R_{t(1,\alpha)}$ on $\mathbb{T}^{n+1}$ is irrational.

Given $\phi \in \mathcal{C}^r(\mathbb{T}^{n+1}, \mathbb{R}_+^*)$, $r \geq 1$, we define the *reparametrization*, or smooth time change, of $R_{t(1,\alpha)}$ with *speed* $\frac{1}{\phi}$ to be the flow given by

$$\frac{d\theta}{dt} = \frac{\alpha}{\phi(\theta, s)}, \quad \frac{ds}{dt} = \frac{1}{\phi(\theta, s)},$$

where $\theta \in \mathbb{T}^n$ and $s \in \mathbb{T}^1$. The reparametrized flow is still minimal and uniquely ergodic (the invariant measure is $\phi(x)dx$, where dx denotes the Haar measure on $\mathbb{T}^{n+1}$) while more subtle asymptotic properties may change under time change as we will see at the end of this Lecture. Considering a cross–section, $R_{t(1,\alpha)}$ can be viewed as the time 1 suspension over R_α. In the same way, the reparametrized flow can be represented as a *special flow* over R_α, with a roof function φ having the same regularity as ϕ. A special flow over a map f of $\mathbb{T}^{n-1}$ is defined on the manifold obtained from $\{(t, y) \mid y \in T^{n-1}, t \in \mathbb{R}, 0 \leq t \leq \varphi(y)\} \subset \mathbb{R} \times \mathbb{T}^{n-1}$

after identifying pairs $(\varphi(y), y)$ and $(0, f(y))$. Of course when f is a translation the result is again the n–dimensional torus but the vertical vector field $\frac{\partial}{\partial t}$ now induces the reparametrized flow instead of the linear one.

Many of the basic questions concerning the relation between the linear flow and the reparametrized one were raised by Kolmogorov in his I.C.M. address in 1954 [K].

Definition 3.15 *Let $m \geq 1$. A vector $\alpha \in \mathbb{R}^m$ is* diophantine *if the flow built over the translation R_α on $\mathbb{T}^m$ under any strictly positive $\mathcal{C}^\infty$ function $\varphi : \mathbb{T}^m \to (0, +\infty)$ is $\mathcal{C}^\infty$ conjugate to the suspension flow over R_α under the constant function $\hat{\varphi}_0 = \int_{\mathbb{T}^m} \varphi$.*

Equivalently we can say that α is diophantine if and only if any smooth reparametrization of $R_{t(1,\alpha)}$ is $\mathcal{C}^\infty$ conjugate to a linear flow.

Theorem 3.16 *A vector $\alpha \in \mathbb{R}^m$ is* diophantine *if and only if there exist two constants $\gamma > 0$ and $\tau \geq m$ such that*

$$|\alpha \cdot k + p| \geq \gamma(|k| + |p|)^{-\tau} \ \forall k \in \mathbb{Z}^m \setminus \{0\} \ and \forall p \in \mathbb{Z}, \tag{3.3}$$

where $k = (k_1, \ldots k_m)$, $|k| = |k_1| + \ldots + |k_m|$.

Proof. The special flow (or reparametrized flow) is smoothly conjugate to a linear flow if it admits a smooth cross–section for which the return time is constant. Looking for this section as a graph $t = \tau(y)$ we obtain from the definition that α is diophantine if and only if the equation

$$\tau(y + \beta) - \tau(y) = \hat{\varphi}_0 - \varphi(y) \ . \tag{3.4}$$

has a $\mathcal{C}^\infty$ solution τ for any given $\mathcal{C}^\infty$ function φ.

Let $\tau(y) = \sum_{k \in \mathbb{Z}^m} \hat{\tau}_k e^{2\pi i k \cdot y}$, $\varphi(y) = \sum_{k \in \mathbb{Z}^m} \hat{\varphi}_k e^{2\pi i k \cdot y}$. The comparing the Fourier coefficients on both sides of (3.4) one has

$$(e^{2\pi i k \cdot \beta} - 1)\hat{\tau}_k = \hat{\varphi}_0 \delta_{k,0} - \hat{\varphi}_k \ . \tag{3.5}$$

The Fourier coefficients of φ are completely arbitrary (except for the constraint of being rapidly decreasing as $|k| \to \infty$) thus (3.5) has a $\mathcal{C}^\infty$ solution if and only if $|e^{2\pi i k \cdot \beta} - 1|^{-1}$ grows at most as a power of $|k|$ as $|k| \to \infty$, i.e. (3.3). □

Considering time–reparametrizations of flows leads to consider the general constant coefficients linear partial differential equation on $\mathbb{T}^n$

$$D_\alpha u := \alpha \cdot \partial u = v \ , \tag{3.6}$$

where $\alpha \in \mathbb{R}^n$, $\partial u = (\partial_1 u, \ldots, \partial_n u)$, is the gradient of u, $v \in \mathcal{C}^{0,\infty}(\mathbb{T}^n, \mathbb{R}^m)$ (i.e. $v \in \mathcal{C}^\infty(\mathbb{T}^n, \mathbb{R}^m)$ *and* $\int_{\mathbb{T}^n} v(x)dx = 0$). Indeed (3.4) is just the discrete analogue of (3.6) obtained replacing the directional derivative $\alpha \cdot \partial$ with a first order finite difference.

Exercise 3.17 Prove that D_α is hypoelliptic [1] if and only if α is a diophantine vector : $|k \cdot \alpha| \geq \gamma |k|^{-\tau}$ for all $k \in \mathbb{Z}^n \setminus \{0\}$.

Exercise 3.18 Prove that almost all $\alpha \in \mathbb{R}^n$ is diophantine of exponent $\tau > n$.

Exercise 3.19 Have a look to the book of Y. Meyer [Me]. Among many interesting things one finds the following theorem (Proposition 2, p. 16) : Let $\mathcal{R}$ be a real algebraic number field and let n be its degree over $\mathbb{Q}$. Let σ be the $\mathbb{Q}$–isomorphism of $\mathcal{R}$ such that $\sigma(\mathcal{R}) \subset \mathbb{R}$ and let $\alpha_1, \ldots \alpha_n$ be any basis of $\mathcal{R}$ over $\mathbb{Q}$. Then $(\sigma(\alpha_1), \ldots, \sigma(\alpha_n)) \in \mathbb{R}^n$ is diophantine of exponent $\tau = n - 1$. Try to prove it if you remember a tiny bit of Galois theory. Apply it to $(1, \sqrt{2}, \sqrt{3}, \sqrt{6})$ and $(1, 2^{1/3}, 2^{2/3})$.

Exercise 3.20 *Show that if u is a solution of (3.6), $\| \ \|_k$ denotes the $\mathcal{C}^k$ norm, α is diophantine with exponent τ then for all $r > \tau + n - 1$ and for all $i \in \mathbb{N}$ there exists a positive constant A_i such that $\|u\|_i \leq A_i \|v\|_{i+r}$.*

The fact that one needs r more derivatives to bound the norms of u in terms of those of v is what is called the "loss of differentiability". This is not an artefact of the methods used but a concrete manifestation of the unboundedness of the linear operator D_α^{-1}. The main consequence of this fact is that one cannot use Banach spaces techniques to study semilinear equations like $D_\alpha u = v + \varepsilon f(u)$, where ε is some small parameter. These semilinear equations are however typical of perturbation theory.

[1] A constant coefficients linear partial differential operator P is hypoelliptic if all u such that $Pu = v$ are $\mathcal{C}^\infty$ on all open sets where v is $\mathcal{C}^\infty$ (see [H2], p.109).

Exercise 3.21 *Analytic functions on $\mathbb{T}^n$ have exponentially decaying Fourier coefficients. What is the optimal diophantine condition to impose in order to solve (3.4)? What happens in the Gevrey–s case (then one has the bound $|\hat{\varphi}_k| \leq Me^{-a|k|^{1/s}}$)?*

When an irrational flow is reparametrized only the most robust of its asymptotic properties (like ergodicity, topological transitivity, minimality, the vanishing of its topological entropy) are preserved. Other important properties, studied by ergodic theory (see Lecture 10) can be sensitive to time change. However when the time reparametrization function is a coboundary, i.e. (3.6) (or (3.4)) has a regular solution, the reparametrized flow is conjugate to the initial flow. This is always the case as we have seen for diophantine frequencies. It is easy to see (Exercise 3.18) that the set of diophantine numbers has full Lebesgue measure. However the latter set is meagre in the sense of Baire category and the numbers that are not diophantine, the so called Liouvillean numbers, are therefore abundant from the topological point of view.
For a Liouvillean α the reparametrization of $R_{t(1,\alpha)}$ can have asymptotic properties that are much different from the initial flow. For instance the reparametrized flow can be weakly mixing, i.e. has no eigenfunctions at all. Specifically, M.D. Šklover proved existence of analytic weakly mixing reparametrizations for some Liouvillean linear flows on $\mathbb{T}^2$; his result for special flows on which this is based is optimal in that he showed that for any analytic roof function φ other than a trigonometric polynomial there is α such that the special flow under the rotation R_α with the roof function φ is weakly mixing. At about the same time A. Katok found a general criterion for weak mixing. B. Fayad [Fay] showed that for any Liouville translation R_α on the torus $\mathbb{T}^n$ the special flow under a generic $\mathcal{C}^\infty$ function φ is weak–mixing. Still a linear flow of $\mathbb{T}^2$ cannot become mixing under smooth time change, not even under a Lipschitz one [Ko]. The argument is based on Denjoy–Koksma type estimates which fail in higher dimension. Indeed, Fayad showed that there exist $\alpha \in \mathbb{R}^2$ and analytic functions φ for which the special flow over the translation R_α and under the function φ is mixing.

Lecture 4. Hyperbolic systems

The concept of hyperbolicity was introduced by Smale [Sm] and the theory of hyperbolic systems is one of the most important achievements of the modern theory of dynamical systems, with outstanding contributions of the Russian school (Anosov, Arnold, Sinai, etc.) and many others (Bowen, Palis, Pugh, Robbin, Robinson, Shub, etc.). The subject is far too developed to be even briefly illustrated in one Lecture but at the same time it is too important to be ignored. Only some of the most fundamental results are presented here.

4.1 Linear hyperbolic maps. Grobman–Hartman Theorem and Local Structural Stability

Our presentation here will follow very closely Yoccoz's lectures [Yo3].

Let E be a Banach space and $T : E \to E$ be a continuous linear map. The *spectrum* $\mathrm{sp}(T)$ of T is the set of complex numbers $\lambda \in \mathbb{C}$ such that $T_{\mathbb{C}} - \lambda$ is not an automorphism of $E_{\mathbb{C}}$ (here $E_{\mathbb{C}}, T_{\mathbb{C}}$ denote the complexifications of E, T). The spectrum is compact and non empty.

Definition 4.1 *Let $0 < \kappa_s < 1 < \kappa_u$. We say that T is* $\kappa_s - \kappa_u$ hyperbolic *if*

$$sp(T) \cap \{\kappa_s < |z| < \kappa_u\} = \emptyset .$$

We say that T is hyperbolic *if*

$$sp(T) \cap \{|z| = 1\} = \emptyset .$$

Of course if T is hyperbolic then there exists $0 < \kappa_s < 1 < \kappa_u$ such that T is $\kappa_s - \kappa_u$ hyperbolic.

Remark 4.2 In a finite–dimensional space, the eigenvalues of a linear operator T depend continuously on T : thus the hyperbolic elements of the general linear group $\mathrm{GL}(E)$ are open and dense. In the infinite–dimensional case the hyperbolic elements form an open set since the spectrum $\mathrm{sp}(T)$ of a continuous linear map is an upper semincontinuous function of T (see [Ka], p. 208) : for all T and $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|S - T\| < \delta$ then $\sup_{\lambda \in \mathrm{sp}(S)} \mathrm{dist}(\lambda, \mathrm{sp}(T)) < \varepsilon$.

Exercise 4.3 (IMPORTANT !) Show that if T is $\kappa_s - \kappa_u$ hyperbolic then there exists a T–invariant splitting $E = E_s \oplus E_u$ of E and constants $c > 0$, $\kappa_s' < \kappa_s$,

$\kappa_u' > \kappa_u$ such that for all $n \geq 0$ one has

$$\|(T|_{E_s})^n\| \leq c(\kappa_s')^n \ , \quad \|(T|_{E_u})^{-n}\| \leq c(\kappa_u')^{-n} \ .$$

Conversely show that a continuous linear map admitting an invariant splitting with all the properties above is $\kappa_s - \kappa_u$ hyperbolic.

E_s and E_u are called the *stable* (resp. *unstable*) *subspace* of T.

Clearly T leaves 0 fixed. In general if F is a $\mathcal{C}^1$ diffeomorphism of E with a fixed point at 0 and $T = F'(0)$ then the fixed point is *hyperbolic* if T is hyperbolic. It is a *sink* if the spectrum of T is contained inside the unit disk, a *source* if it is contained outside the closed unit disk and a *saddle* otherwise.

The properties of T and its iterates are best studied using adapted norms.

Definition 4.4 *A norm* $|\ |$ *on* E *is* adapted *to* T *if : (i)* $|\ |$ *and* $\|\ \|$ *are equivalent ; (ii)* $|x_s + x_u| = \max(|x_s|, |x_u|)$ *when* $x_s \in E_s$, $x_u \in E_u$ *; (iii)* $|(T|_{E_s})| \leq \kappa_s$ *and* $|(T|_{E_u})^{-1}| \leq \kappa_u^{-1}$.

Exercise 4.5 Show that adapted norms always exist. [Hint : use the splitting of E and define them for points in E_s and E_u, then impose (ii)].

Using the contracting maps fixed point theorem one can prove the following important theorem (see Appendix 3 for its proof) :

Theorem 4.6 (Grobman–Hartman) *Let T be a linear $\kappa_s - \kappa_u$ hyperbolic automorphism of E and $|\ |$ be an adapted norm. Suppose that $F = T + \Delta F$, where ΔF is bounded and Lipschitz. If the Lipschitz constant ε of ΔF is such that*

$$\varepsilon < \min(|T^{-1}|^{-1}, 1 - \kappa_s, 1 - \kappa_u^{-1}) \ .$$

Then there exists a unique homeomorphism $h = id + \Delta h$ such that Δh is bounded and which conjugates F to T.

4.2 Hyperbolic compact invariant sets. Axiom A systems

Let K be a compact metric space and $\pi : E \to K$ a finite–dimensional vector bundle over K. For $x \in K$ we denote E_x the fiber $\pi^{-1}(x)$. We choose once and for all some continuous family of norms $(\|\ \|_x)_{x \in E_x}$.

Let f be a homeomorphism of K.

Definition 4.7 *A linear automorphism F of E is* fibered over f *if for each $x \in K$ the restriction F_x of F to E_x is a linear isomorphism from E_x onto $E_{f(x)}$ which depends continuously on x.*

We denote $\Gamma(E)$ the Banach space of continuous sections of π. Given a linear automorphism F fibered over f we get a linear map $\mathcal{F}$ over $\Gamma(E)$ defining for a section $\sigma \in \Gamma(E)$

$$\mathcal{F}(\sigma)(f(x)) = F_x(\sigma(x)) \ .$$

Proposition 4.8 *The linear map $\mathcal{F}$ is $\kappa_s - \kappa_u$–hyperbolic if and only if there exists an F–invariant splitting $E = E_s \oplus E_u$ into continuous subbundles of E and constants $c > 0$, $\kappa_s' < \kappa_s$, $\kappa_u' > \kappa_u$ such that for all $n \geq 0$ and for all $x \in K$ one has*

$$\|(F|_{E_{s,x}})^n\| \leq c(\kappa_s')^n \ , \quad \|(F|_{E_{u,x}})^{-n}\| \leq c(\kappa_u')^{-n} \ .$$

*We then say that F is $\kappa_s - \kappa_u$–*hyperbolic *and that E_s (resp. E_u) is the* stable *(resp.* unstable*) subbundle of F.*

For the proof of the previous proposition see [Yo3].

Definition 4.9 *Let X be a smooth manifold equipped with some Riemannian metric and $f \in Diff^1(X)$. A compact f–invariant subset K of X is* hyperbolic *if the restriction of the tangent map Df to $TX|_K$ is hyperbolic in the sense of Proposition 4.8. If $K = X$ then f is an* Anosov system.

Remark 4.10 If f is an endomorphism of X one requires $Df|_{E_u}$ to be invertible. For a flow $(\varphi_t)_{t\in\mathbb{R}}$ one must add to the splitting a one–dimensional subbundle E_0 in the direction of the flow $TX|_K = E_s \oplus E_u \oplus E_0$ and require for some Riemannian metric and some constants $C > 0$ and $0 < \lambda < 1$ that $\max(\|D\varphi_t|_{E_s}\|, \|d\varphi_{-t}|_{E_u}\|) \leq Ce^{\lambda t}$ for all $t \in \mathbb{R}$.

Hyperbolic linear are Anosov, as well as hyperbolic toral automorphisms (see Exercises 4.21–4.25) and geodesic flows on compact Riemannian manifolds with negative curvature. Anosov systems are structurally stable [An, Mat1]. On tori they are all "given" by linear automorphisms [Man2] since they are all topologically conjugate to them. The compact invariant set K does not need to be connected (for example many attractors may coexist) thus the fundamental concept in the

theory of hyperbolic systems is that of a *basic set*. If f is a diffeomorphism a basic set is a *compact, invariant* subset K of X that is *transitive* (i.e. $f|_K$ has a dense orbit), *locally maximal* (i.e. K is the maximal invariant set in an open neighborhood) and *hyperbolic*.

Definition 4.11 *The* chain recurrent set $C(f)$ *of a diffeomorphism of a compact manifold X is the locus of points that are periodic for some arbitrarily small $\mathcal{C}^0$–perturbation of f. A* uniformly hyperbolic *diffeomorphism is an f with $C(f)$ with a hyperbolic splitting.*

Exercise 4.12 Show that $C(f)$ is closed and f–invariant.

One can prove that for a uniformly hyperbolic diffeomorphism $C(f)$ is a finite union of disjoint basic sets. Uniformly hyperbolic diffeomorphisms form an open subset of $\mathrm{Diff}^\infty(X)$ and they are *stable* under $\mathcal{C}^1$ perturbations [Ro, Rob] : two $\mathcal{C}^1$ close uniformly hyperbolic diffeomorphisms are topologically conjugated on a neighborhood of their respective chain recurrent sets. A very deep result of Mañe [Mn2] states that the converse is also true : a diffeomorphism which is $\mathcal{C}^1$ stable in the above sense is also uniformly hyperbolic. This result has been later extended to flows by Hayashi [Hay]. The results of Mañe and Hayashi show that uniformly hyperbolic systems are, essentially, the structurally stable ones.

Definition 4.13 *Let K be a compact invariant set. We say that K is an* Axiom A *attractor if : (i) K is hyperbolic ; (ii) there exists a compact neighborhood U of K such that $f(U) \subset U$ and $K = \cap_{n\geq 0} f^n(U)$; (iii) $f|_K$ is topologically transitive.*

4.3 Expansivity and Shadowing

Perhaps the two most important features of the dynamics of hyperbolic systems are the *shadowing property* and the *expansivity property*.

Definition 4.14 *Let K be a compact metric space and f a homeomorphism of K. Let $\delta > 0$. A* δ–pseudo orbit *is a sequence $\{x_i\}_{i\in\mathbb{Z}} \subset K$ such that $d(f(x_i), x_{i+1}) \leq \delta$ for all $i \in \mathbb{Z}$.*

Exercise 4.15 Show that a point x belongs to the chain recurrent set $C(f)$ of

f if and only if for all $\delta > 0$ there exists a periodic δ–pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ with $x_0 = x$.

Exercise 4.16 Show that all non wandering points belong to the chain recurrent set.

Let $f \in \mathrm{Diff}^1(X)$ and K be a compact invariant hyperbolic subset of X.

Theorem 4.17 (Shadowing Lemma) *There exists a neighborhood U of K and a constant $\theta > 0$ such that for any $\delta > 0$ and any δ–pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset U$ there exists $x \in X$ such that $d(f^i(x), x_i) \leq \theta\delta$ for all $i \in \mathbb{Z}$.*

Theorem 4.18 (Expansivity) *There exists $\varepsilon > 0$ and a neighborhood U of K in X such that if x, y are two distinct points in $\cap_{n\in\mathbb{Z}} f^{-n}(U)$ we have $\sup_{n\in\mathbb{Z}} d(f^n(x), f^n(y)) \geq \varepsilon$.*

Remark 4.19 Note that if δ and U in 4.17 are small enough then the point x is unique by the expansivity property.

4.4 Bernoulli shifts, hyperbolic automorphisms of the torus and geodesic flows on manifolds with negative curvature

Let $X = \{0,1\}^{\mathbb{Z}}$, $x = \{x_i\}_{i\in\mathbb{Z}} \in X$, $y = \{y_i\}_{i\in\mathbb{Z}} \in X$ and let d denote the metric on X defined by $d(x,y) = \sum_{i\in\mathbb{Z}} |x_i - y_i| 2^{-|i|}$. X is compact and totally disconnected, homeomorphic to the usual Cantor set.

Let f denote the shift on X : $(f(x))_i = x_{i+1}$. Clearly f is a continuous topologically transitive homeomorphism of X. It is called the *Bernoulli shift* on two symbols.

Exercise 4.20 Show that the Bernoulli shift has the expansivity property. Moreover given $x \in X$ and $\varepsilon > 0$ consider the stable, local stable, unstable and local unstable sets :

$$
\begin{aligned}
W_s(x) &= \{y \in X \mid d(f^n(y), f^n(x)) \to 0 \text{ as } n \to +\infty\}\ , \\
W_{s,\varepsilon}(x) &= \{y \in W_s(X) \mid d(f^n(y), f^n(x)) \leq \varepsilon\ n \geq 0\}\ , \\
W_u(x) &= \{y \in X \mid d(f^n(y), f^n(x)) \to 0 \text{ as } n \to -\infty\}\ , \\
W_{u,\varepsilon}(x) &= \{y \in W_u(X) \mid d(f^n(y), f^n(x)) \leq \varepsilon\ n \leq 0\}\ .
\end{aligned}
$$

Determine them and show that there are $\delta > 0$ and $\varepsilon > 0$ so that for each pair of points $x, y \in X$ with $d(x,y) < \delta$, $W_{s,\varepsilon}(x) \cap W_{u,\varepsilon}(y)$ is a single point (in this case we say that an expanding f has a *local product structure*).

Let us now consider the continuous automorphisms of the n–dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Exercise 4.21 Prove that the group of continuous automorphisms of $\mathbb{T}^n$ is isomorphic to $\mathrm{GL}(n,\mathbb{Z})$ (the group of $n \times n$ matrices with integer coefficients and determinant ± 1).

$A \in \mathrm{GL}(n,\mathbb{Z})$ is hyperbolic if and only if it has no eigenvalue of modulus one. Let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the canonical projection.

Exercise 4.22 Let $A \in \mathrm{GL}(n,\mathbb{Z})$ be hyperbolic. Construct the hyperbolic A–invariant splitting of $\mathbb{R}^n$ considering the Jordan form of A and using the Jordan blocks with diagonal entries of modulus less than one to construct E_s and those with diagonal entries of modulus greater than one to construct E_u. Prove that $W_s(0) = \pi(E_s)$ and $W_u(0) = \pi(E_u)$ are one–to–one immersed copies of R^s and R^u and each of them is dense in $\mathbb{T}^n$.

Exercise 4.23 Prove that an automorphism A of the torus $\mathbb{T}^n$ has the expansivity property if and only if it is hyperbolic. [Hint : note that if A has an eigenvalue of modulus 1 then there is a two–dimensional subspace of $\mathbb{R}^n$ with points arbitrarily near zero whose distance from zero is bounded under forward and backward iterates of A.]

Exercise 4.24 Prove that a point $x \in \mathbb{T}^n$ has a periodic orbit under a hyperbolic automorphism A if and only if all its coordinates are rational. Use this fact to prove that $N_m = \sharp\{A^m(x) = x\} = |\det(A^m - I)|$. Conclude that the Artin–Mazur ζ–function of A is rational.

Exercise 4.25 Prove that two automorphisms A, B of the torus are topologically conjugate if and only if they are conjugate in $\mathrm{GL}(n,\mathbb{Z})$.

We conclude this very short survey of hyperbolic systems with the most important example of hyperbolic flow : the geodesic flow on a surface with constant negative curvature.

The upper half–plane model of two–dimensional hyperbolic space is $\mathbb{H} = \{z \in \mathbb{C} \mid \Im m\, z > 0\}$ equipped with the Riemannian metric $(ds)^2 = \frac{dz d\overline{z}}{(\Im m\, z)^2}$. The

boundary of $\mathbb{H}$ is $\mathbb{R} \cup \{\infty\}$, the metric has constant Gaussian curvature -1 and d denotes the associated distance. The isometries group of $\mathbb{H}$ is $\mathrm{PSL}(2,\mathbb{R})$ acting on $\mathbb{H}$ by homographies : $Az = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$. Note that $(Az)' = 1/(cz+d)^2$.

A unit tangent vector $v \in \mathbb{T}_z\mathbb{H}$ determines a geodesic $\gamma_v : \mathbb{R} \to \mathbb{H}$ with $v = \dot{\gamma_v}(0)$. Here $\gamma_v(\mathbb{R})$ is either the Euclidean semicircle meeting $\partial\mathbb{H}$ orthogonally at two points and passing through z with velocity v or the vertical line through z if v has no horizontal component.

If X is a complete surface with Riemannian metric of constant curvature -1 and finite area then the universal cover of X is $\mathbb{H}$. The fundamental group of X acts on $\mathbb{H}$ by a discrete group Γ of hyperbolic isometries and $X = \mathbb{H}/\Gamma$. Since geodesics are preserved by Γ then they pass to the quotient and they give the geodesics on X.

The *geodesic flow* g_t on X is defined on the unit tangent bundle UX of X considering the time evolution of a geodesic $\gamma_v(t)$ together with the time evolution of its velocity $v(t) \in U_{\gamma_v(t)}X$. The flow is the projection to UX of the geodesic flow on $U\mathbb{H}$.

The isometries group $\mathrm{PSL}(2,\mathbb{R})$ acts transitively on $U\mathbb{H}$ thus identifying $U\mathbb{H}$ with $\mathrm{PSL}(2,\mathbb{R})$ one can easily express the geodesic flow on $U\mathbb{H}$. Let us see how. Consider the geodesic γ_{i_i} determined by the unit vector i_i pointing vertically upwards and based at $i \in \mathbb{H}$. Then $\gamma_{i_i}(\mathbb{R})$ is the imaginary axis parametrized by $\gamma_{i_i}(t) = ie^t$. The isometry $g(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ sends i to $\gamma_{i_i}(t)$ preserving the imaginary axis. Now we identify $\mathrm{PSL}(2,\mathbb{R})$ with $U\mathbb{H}$ associating to $A \in \mathrm{PSL}(2,\mathbb{R})$ the unique unit tangent vector $Ai_i \in U_{Ai}\mathbb{H}$: B will send the geodesic γ_{i_i} to a geodesic γ_{Ai_i} and the g_t–orbit $\left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, t \in \mathbb{R} \right\}$ to $\left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} A, t \in \mathbb{R} \right\}$. Thus the geodesic flow g_t , $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ is given by $g_t(A) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} A$.

The finite area surface $X = \mathbb{H}/\Gamma$, with Γ a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, the unit tangent bundle $UX = \mathrm{PSL}(2,\mathbb{R})/\Gamma = \{A\Gamma, A \in \mathrm{PSL}(2,\mathbb{R})\}$ and the geodesic flow is now given by $g_t(\Gamma A) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} A\Gamma$.

Exercise 4.26 Show that the Liouville measure on $\mathrm{PSL}(2,\mathbb{R})$, i.e. the product measure of the area on $\mathbb{H}$ and the length on the unit circle of unit vectors at any

given point is g_t–invariant.

Exercise 4.27 Show that the hyperbolic circle $C(z,r)$ of center z and radius r is a Euclidean circle in $\mathbb{H}$ (although the center is not z). [Hint : use the isometry that sends $\mathbb{H}$ to the Poincaré disk $\mathbb{D}$ and z to 0.]

Given $v \in U_z\mathbb{H}$ the circle $C(\gamma_v(r), r)$ converges, as $r \to \infty$, either to a Euclidean circle touching $\partial\mathbb{H}$ at the positive end of γ_v and having v as the inward normal at z or to the half–plane above the horizontal line through z. In both cases this is called the positive horosphere $S^+(v)$. Similarly $C(\gamma_v(-r), r)$ converges to the negative horosphere $S^-(v)$ (see Fig. 4.1). All the inward–pointing (resp. outward–pointing) normals to $S^+(v)$ (resp. $S^-(v)$) define geodesics with the same ω–limit (resp. α–limit) set of γ_v. Quotienting out by the action of Γ one obtains the horospheres for a unit tangent vector in UX.

We define the horocycle flows $h_t\,,\, h_t^* \;:\; UX \to UX$ as the flows related by $h_t^*(v) = -h_{-t}(-v)$ with h_t^* which slides the inward normal vectors to $S^+(v)$ to the right along $S^+(v)$ at unit speed.

Exercise 4.28 Show that under the identification $UX = \mathrm{PSL}\,(2,\mathbb{R})/\Gamma$ the horocycle flows are given by multiplications on the left by $h_t^* = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $h_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. Show that the Liouville measure (Exercise 4.26) is invariant under h_t and h_t^*.

At each point $v \in UX$ we have the three–dimensional tangent space T_vUX with a basis given by the vectors tangent, at $t = 0$, to the curves in UX given by g_tv, h_tv and h_t^*v. The three one–dimensional subspaces spanned by these tangent

vectors will be called $E_{0,v}$, $E_{u,v}$ and $E_{s,v}$ respectively. They are invariant under the geodesic flow in the sense that $(g_t)_* E_{a,v} = E_{a,g_t v}$, with $a = 0, u, v$. Thus we get a splitting of the tangent bundle to $UX = E_0 \oplus E_u \oplus E_s$. Since one has the relations

$$g_t g_s = g_s g_t \ ,$$
$$g_t h_s = h_{se^t} g_t \ , \text{ negative horocycle flow expands ,}$$
$$g_t h^*_s = h^*_{se^{-t}} g_t \ , \text{ positive horocycle flow contracts ,}$$

one gets

$$\|(g_t)_*|_{E_{0,v}}\| = 1 \ , \ \ \|(g_t)_*|_{E_{u,v}}\| = e^t \ , \ \ \|(g_t)_*|_{E_{s,v}}\| = e^{-t} \ ,$$

which show that the geodesic flow is hyperbolic.

Part II. Important examples. Renormalization

Lecture 5. Holomorphic Dynamics in One Dimension

In this Lecture we begin the study of the dynamics of the endomorphisms of the Riemann sphere. Some useful background material from complex analysis is recalled in Appendix 4.

5.1 Endomorphisms of the Riemann Sphere

Let us first of all recall the notion of normal family on an open subset U of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. To this purpose we recall the usual system of coordinates on $\overline{\mathbb{C}}$ determined by the stereographic projection : $z : \overline{\mathbb{C}} \setminus \{\infty\} \to \mathbb{C}$, $z(0) = 0$ and $w : \overline{\mathbb{C}} \setminus \{0\} \to \mathbb{C}$, $w(\infty) = 0$, related by $zw = 1$. The *spherical metric* $d_{\overline{\mathbb{C}}}$ on $\overline{\mathbb{C}}$ is defined as follows :

$$(ds_{\overline{\mathbb{C}}})^2 = \begin{cases} \frac{(dz)(d\overline{z})}{\pi(1+|z|^2)^2} & \text{in the } z\text{–chart,} \\ \frac{(dw)(d\overline{w})}{\pi(1+|w|^2)^2} & \text{in the } w\text{–chart,} \end{cases} \tag{5.1}$$

where the normalization is such that area $(\overline{\mathbb{C}}) = 1$. Let $U \subset \overline{\mathbb{C}}$ be open. A meromorphic function on U is an analytic map $f : U \to \overline{\mathbb{C}}$ and $\mathcal{F}_U = \{f : U \to \overline{\mathbb{C}},\ f \text{ analytic}\}$. We endow $\overline{\mathbb{C}}$ with the spherical metric and $\mathcal{F}_U$ with the topology of uniform convergence on compact subsets of U. It is a classical result of Weierstrass that the limit of a convergent sequence in $\mathcal{F}_U$ still belongs to $\mathcal{F}_U$ (note that the constant function $f \equiv \infty$ is considered meromorphic).

Definition 5.1 *A family $\mathcal{F} \subset \mathcal{F}_U$ is* normal *if it is relatively compact in $\mathcal{F}_U$, i.e. any sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges uniformly in the spherical metric on compact subsets of U.*

Warning ! If $\{f_n\}$ is a normal family then $\{f_n'\}$ needs not be normal : e.g. $f_n(z) = n(z^2 - n)$ on $\mathbb{C}$.

By means of the Ascoli–Arzelà theorem and of Cauchy's integral theorem one gets :

Proposition 5.2

(I) A family of meromorphic functions on U is normal on U if and only if it is equicontinuous on every compact subset of U ;

(II) A family of analytic functions on U is normal on U if and only if it is locally uniformly bounded (i.e. uniformly bounded on every compact subset of U).

Exercise 5.3 Prove Montel's Theorem : every family of meromorphic functions on some domain of $\mathbb{C}$ which omits three fixed values is normal

The notion of normal family allows us to introduce the basic notions of one–dimensional holomorphic dynamics. Here we are interested in studying the dynamics of a discrete dynamical system (i.e. an action of $\mathbb{N}$) on the Riemann sphere $\overline{\mathbb{C}}$ generated by a holomorphic transformation $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, i.e. and element of $\mathrm{End}\,(\overline{\mathbb{C}})$.

Let d denote the topological degree of R (i.e. the number of preimages of a typical point z) : R can be written in a unique way in the form $R(z) = \frac{P(z)}{Q(z)}$, where $P(z) \in \mathbb{C}[z]$, $Q(z) \in \mathbb{C}[z]$ have no common factors and $d = \max(\deg P, \deg Q)$. In fact every $d : 1$ conformal branched covering of $\overline{\mathbb{C}}$ comes from some such rational function and

$$\mathrm{End}\,(\overline{\mathbb{C}}) = \{R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}\,\text{holomorphic}\} = \{R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}\,,\, R(z) = P(z)/Q(z)\}\ .$$

Note that

$$R(z) = \begin{cases} \frac{P(z)}{Q(z)} & \text{if } Q(z) \neq 0, \\ \infty & \text{if } Q(z) = 0, \\ \lim_{z\to\infty} \frac{P(z)}{Q(z)} & \text{if } z = \infty. \end{cases}$$

We define the iterates R^n of R as usual : $R^n = R \circ R^{n-1}$. Note that R^n has degree d^n.

The dynamics in the case $d = 1$ is trivial : $R \in \mathrm{PGL}\,(2, \mathbb{C}) = \mathrm{Aut}\,(\mathbb{C})$ and one has the classical distinction among hyperbolic, parabolic and elliptic Möbius transformations :

- *hyperbolic elements* A have $|\mathrm{tr}\,(A)| > 2$, two distinct fixed points and they are conjugate in $\mathrm{Aut}\,(\overline{\mathbb{C}})$ to $z \mapsto \lambda z$ with $0 < \lambda < 1$;
- *parabolic elements* have $|\mathrm{tr}\,(A)| = 2$, one fixed point and they are conjugate in $\mathrm{Aut}\,(\overline{\mathbb{C}})$ to $z \mapsto z + 1$;
- elliptic elements have $|\mathrm{tr}\,(A)| < 2$, two distinct fixed points and they are conjugate in $\mathrm{Aut}\,(\overline{\mathbb{C}})$ to $z \mapsto \lambda z$ with $|\lambda| = 1$.

We will assume $d \geq 2$: in this case any rational map has both expanding and contracting features. Since R is a d–fold branched covering of the Riemann sphere it must be expanding on average. Indeed if $d\sigma_{\overline{\mathbb{C}}}$ is the area 2–form on $\overline{\mathbb{C}}$ associated to the spherical metric (5.1)

$$\int_{\overline{\mathbb{C}}} \|(R^n)'\|^2 d\sigma_{\overline{\mathbb{C}}} = \frac{i}{2\pi} \int_{\overline{\mathbb{C}}} \frac{|(R^n)'(z)|^2 dz \wedge d\overline{z}}{(1+|R^n(z)|^2)^2} = d^n \to \infty \ ,$$

so the derivative of R^n is very large on average. On the other hand, by the Riemann–Hurwitz formula, R has $2d-2$ *critical points* c where $R'(c)=0$. Near a critical point R is highly contracting since its behaviour is like that of $z \mapsto z^k$ near the origin, for some $k > 1$. The *postcritical set* $P(R)$ is the closure of the forward orbits of the critical points of R :

$$P(R) = \overline{\cup_{n>0\,,\,R'(c)=0} R^n(c)} \ .$$

The degree of R also determines its topological entropy :

Theorem 5.4 *The topological entropy of R is equal to* $\log d$.

The proof of Theorem 5.4 is obtained in two steps : first one proves (see [MP]) that given any $\mathcal{C}^1$ map f of a compact smooth orientable manifold X the topological entropy of f is at least $\log|\deg(f)|$, where $\deg(f)$ is the topological degree of f. Then in the special case of analytic endomorphisms of the Riemann sphere one can prove that the degree gives an upper bound too (see [Ly1]). Note that the topological entropy is equal to the exponential growth rate of the number of periodic orbits of R.

Given a point $z_0 \in \overline{\mathbb{C}}$ its orbit is the sequence $\{z_n\}_{n\geq 0}$ defined by $z_{n+1} = R(z_n)$. The point z_0 is called *preperiodic* if, for some $k > 0$, z_k is periodic. Periodic points $R^m(z_0) = z_0$ are classified according to the value λ of the derivative of R^m (the *multiplier*) at z_0

$$|\lambda| = |(R^m)'(z_0)| \begin{cases} > 1 & \text{repelling,} \\ = 1 & \text{indifferent,} \\ < 1 & \text{attracting,} \\ = 0 & \text{superattracting.} \end{cases}$$

Indifferent periodic orbits are *rationally indifferent* (or *parabolic*) if the multiplier is a root of unity, *irrationally indifferent* otherwise.

Let A be an attracting periodic orbit. We have the following :

Proposition 5.5 *Every attracting periodic orbit A attracts a critical point.*

Proof. Let $U = \{z : d_{\overline{\mathbb{C}}}(R^n(z), A) \to 0\}$. U is open, simply connected, and $f^{-1}(U) = U$. If U contains no critical point then $R|_U$ is a covering map and the Schwarz lemma implies that it is an isometry for the hyperbolic metric on U, which is impossible because A is attracting. □

The previous Proposition implies that the number of attracting cycles is bounded by the number of critical points, thus by $2d - 2$.

The competition between the expanding and contracting features of R is responsible for much of the complexity of the dynamics of rational maps.

5.2 Fatou and Julia sets

The fundamental dichotomy of $\overline{\mathbb{C}}$ associated to the dynamics of R is the following :

Definition 5.6 *The* Fatou set $F(R)$ *of R is the set of points $z_0 \in \overline{\mathbb{C}}$ such that $\{R^n\}_{n\geq 0}$ is a normal family in some disk $D(z_0, r)$ (w.r.t. the spherical metric). The complement of the Fatou set is the* Julia set $J(R)$.

Exercise 5.7 Show that

(i) The Fatou set is open and the Julia set is closed.

(ii) The Fatou set and the Julia set are transported by rational semiconjugacy : if $R, S, H \in \mathrm{End}\,(\overline{\mathbb{C}})$ and $R \circ H = H \circ S$ then $J(R) = H(J(S))$.

(iii) If two rational maps commute $R \circ S = S \circ R$ then they have the same Julia set : $J(R) = J(S)$.

(iv) The Fatou set and the Julia set are *completely invariant* : $z \in F(R)$ (resp. $J(R)$) if and only if $R(z) \in F(R)$ (resp. $J(R)$). [Hint : a non constant holomorphic map takes open sets to open sets.]

(v) The Julia set is nonempty. [Hint : if $J(R) = \emptyset$ then $\{R^n\}_{n\geq 0}$ is a normal family on all $\overline{\mathbb{C}}$ thus $R^{n_j} \to S \in \mathrm{End}\,(\overline{\mathbb{C}})$. Compare degrees.]

(vi) The Julia set is nowhere dense or it coincides with the whole Riemann sphere. [Hint : suppose that $J(R)$ contains an open set and apply Montel's theorem.]

(vii) The Julia set is perfect, i.e. it has no isolated points. Apply Baire's Category Theorem [Ox] and deduce that $J(R)$ is uncountable.

(viii) $R|_{J(R)}$ is topologically transitive.

Since the Julia set is completely invariant it possesses a great deal of *self–similarity* : let us make this point more precise. Given two points z, z' of $J(R)$, we say that the Julia set at z is *locally conformally isomorphic* to the Julia set at z' if there exists a conformal isomorphism from a neighborhood U of z to a neighborhood U' of z' which carries z to z' and $J(R) \cap U$ onto $J(R) \cap U'$. Clearly whenever $R^k(z) = z'$ for some $k \geq 1$ with derivative $(R^k)'(z) \neq 0$ then the Julia set at z is locally conformally isomorphic to the Julia set at z'.

Exercise 5.8 Show that for all but finitely many points z of $J(R)$ the set of z' for which the Julia set at z' is locally conformally isomorphic to the Julia set at z is everywhere dense in $J(R)$.

The Julia set can also be equivalently characterized as the *smallest closed subset of the sphere which is completely invariant and contains at least three points* as the following Proposition shows :

Proposition 5.9 *Let E be a closed completely R–invariant subset of $\overline{\mathbb{C}}$ which contains at least three points. Then $J(R) \subset E$.*

Proof. Since E is completely invariant also $D = \overline{\mathbb{C}} \setminus E$ is completely invariant. The family $\{R^n|_D\}_{n\in\mathbb{N}}$ omits three values thus it is normal. Thus $J(R) \subset E$. □

Exercise 5.10 Let $[z]$ denote the smallest completely invariant set which contains z and let $E(R) = \{z \in \overline{\mathbb{C}} \mid [z] \text{ is finite}\}$. Show that $E(R) \subset F(R)$ and has at most two points.

The Julia set of a rational map is the locus of expanding and chaotic behaviour of the dynamics. For example one has the following equivalent characterization of $J(R)$:

Theorem 5.11 *The Julia set $J(R)$ is the closure of the set of repelling periodic orbits.*

Proof. It is immediate to check that if z belongs to the closure of the set of repelling periodic orbits then it also belongs to $J(R)$ (in any neighborhood of z there exists a repelling periodic orbit ; if n denotes its period consider $(R^{nj})'$ as $j \to \infty$).

Conversely assume now that $z \in J(R)$ and suppose that there exists a disk D around z which contains no fixed point of R^n for all $n \in \mathbb{N}$, $n \geq 1$. One can assume that D contains no poles of R nor critical values. Choose two different

branches S_1, S_2 of R^{-1} on D, then $T_n = \frac{(R^n - S_1)(z-S_2)}{(R^n-S_2)(z-S_2)}$ is a family of holomorphic functions on D which omits the values $0, 1, \infty$. By Montel's theorem (Exercise 5.3) the family is normal, and hence so is R^n. This shows that periodic points are dense in $J(R)$. To conclude that repelling periodic points are dense we use the finiteness of the number of attracting cycles (Proposition 5.5) and also of indifferent cycles (this is a little bit more tricky : see [CG], paragraph III.2). □

Since normal families are precompact the dynamics on the Fatou set is regular and predictable. Indeed it is the locus of the *stable* points.

Definition 5.12 *A point $z_0 \in \overline{\mathbb{C}}$ is* stable *if for all $\delta > 0$ there exists a neighborhood W of z_0 such that for all $z \in W$ and for all $n \geq 0$ one has $d_{\overline{\mathbb{C}}}(R^n(z), R^n(z_0)) \leq \delta$.*

Exercise 5.13 Show that a point is stable if and only if it belongs to the Fatou set.

5.3 Examples

Example 5.14 Let $R(z) = z^d$. Then $J(R) = \mathbb{S}^1$ and $R|_{\mathbb{S}^1}$ is topologically transitive and has topological entropy $\log d$. Note that $|R'|_{\mathbb{S}^1}| = d > 1$, all periodic points of $\mathbb{S}^1$ are repelling.

Example 5.15 (Tchebychev Polynomials) The Tchebychev polynomials $T_k \in \mathbb{C}[z]$ have degree k and are defined through the functional equation $T_k(\cos z) = \cos(kz)$. Note that this implies that $T_k^n(\cos z) = \cos(k^n z)$ for all $n = 1, 2, \ldots$. Let $k \geq 2$. Show that $J(T_k) = [-1, 1]$. [Hint : the map $H(z) = (z + 1/z)/2$ (semi)–conjugates T_k with $R(z) = z^k$.]

Example 5.16 Let $R(z) = 2z - 1/z$. The upper and lower half–planes and the exterior of the closed unit disk are invariant under R. Thus by Montel's theorem $J(R) \subset [-1, 1]$. Study the iterates of $R|_{[-1,1]}$ and show that $J(R)$ is a Cantor set. [Hint : First construct the Cantor set K of points $z \in [-1, 1]$ such that $|R^m(z)| \leq 1$ for all $m \in \mathbb{N}$. Then note that $|R'|_{[-1,1]}| \geq 3$ and use the fact that a normal family bounded in absolute value on a compact subset of $\mathbb{C}$ must also have bounded derivative.]

Example 5.17 (Lattès) Let $X = \mathbb{C}/\Lambda$ be a complex torus, where $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$

is a lattice in the complex plane, $\tau \notin \mathbb{R}$. Let $\alpha \in \mathbb{C}$, $|\alpha| > 1$, such that $\alpha\Lambda \subset \Lambda$. Then multiplication by α induces an endomorphism $F : X \to X$. For example one can always choose $\alpha = n$, then F has degree n^2. If $\alpha \notin \mathbb{N}$ then we say that X has a *complex multiplication* : in this case τ is an algebraic number of degree two. Since $|F'(x)| = |\alpha| > 1$ the map F is uniformly expanding and repelling periodic points of F are dense on X. [Hint : all the $|\alpha^k - 1|^2$ points $(m + n\tau)/(\alpha^k - 1)$ of order $|\alpha^k - 1|$ in the group law on X have period k under F]. Show that :

(i) the quotient of X by the equivalence relation $x \approx -x$ is the Riemann sphere. [Hint : compute the Euler characteristic $\chi(X/\approx)$ either directly or using Riemann–Hurwitz.] Note that F commutes with $\approx$.

(ii) The quotient map $\wp : X \to \overline{\mathbb{C}}$ is given by Weierstrass $\wp$–function and presents X as a two–to–one branched cover of $\overline{\mathbb{C}}$. The branching points are the projection under $\wp$ of the four fixed points $0, 1/2, \tau/2$ and $(1 + \tau)/2$ of $x \approx -x$ modulo Λ.

(iii) Let R be the rational map on $\overline{\mathbb{C}}$ induced by $F : X/\approx \to X/\approx$ through $\wp$. The degree of R is $|\alpha|^2$ and $R(\wp(z)) = \wp(\alpha(z))$ (compare with the Tchebychev polynomials).

(iv) Finally $J(R) = \overline{\mathbb{C}}$ since the periodic points of F are repelling and dense in X.

(v) Show that the involution $x \mapsto x + 1/2$ of X corresponds under $\wp$ to the involution $z \mapsto e_3/z$ of $\overline{\mathbb{C}}$, where we have put as it is usual $e_1 = \wp(1/2)$, $e_2 = \wp(\tau/2)$, $e_3 = \wp((1 + \tau)/2)$.

(vi) Fix $\alpha = 2$. Then R has poles at ∞, e_1, e_2 and e_3 and double zeros at $\pm\sqrt{e_3}$. Using the duplication formula for $\wp$ (see [WW, p. 441]) show that $R(z) = \left[z^4 + g_2\frac{z^2}{2} + 2g_3 z + \left(\frac{g_2}{4}\right)^2\right]\left[4z^3 - g_2 z - g_3\right]^{-1}$, where $g_2 = 60\sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-4}$, $g_3 = 140\sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-6}$.

(vii) *R is not rigid* : varying τ we obtain a family of rational maps which are topologically but not conformally conjugate (unless the different values of τ belong to the same orbit under the modular group $\mathrm{PSL}(2, \mathbb{Z})$).

Lecture 6. No wandering domains and classification. Quasiperiodic components. Hyperbolicity conjectures

In this Lecture we will first describe a result due to Dennis Sullivan [Su2] which shows that no component of the Fatou set can be wandering. Then, following Fatou's classification of stable domains, we discuss briefly quasiperiodic orbits. Finally we present the conjecture of density of hyperbolicity for rational maps and the related conjecture of non existence of invariant line fields on the Julia set.

6.1 Sullivan's no wandering domains theorem and the classification of Fatou's components

No wandering point can belong to the Julia set of a rational non invertible endomorphism of the Riemann sphere as the following Exercise shows :

Exercise 6.1 Let U be a non–empty open set which meets $J(R)$. Then for all sufficiently large n one has $J(R) \subset R^n(U)$.

A rational map R maps each component of its Fatou set to another component. A priori however one could not exclude the existence of wandering domains in the Fatou set : a component U of $F(R)$ is *wandering* if the sets $R^n(U)$, $n \geq 0$, are pairwise disjoint. One of the most important achievements of the theory is the proof due to Dennis Sullivan that this cannot occur [Su2].

Theorem 6.2 (No wandering domains and classification) *Every component of the Fatou set is preperiodic. The number of periodic components is finite. A periodic component U (of period p) is of exactly one of the following types :*

(i) *An* attracting basin *: there is an attracting or superattracting periodic point w in U and $R^{np}(z) \to w$ for all $z \in U$ as $n \to \infty$.*

(ii) *A* parabolic basin *: there is a rationally indifferent periodic point $w \in \partial U$ and $R^{np}(z) \to w$ for all $z \in U$ as $n \to \infty$.*

(iii) *A* Siegel disk *: the component U is conformally equivalent to a disk on which R^p is an irrational rotation.*

(iv) *A* Herman ring *: the component U is conformally equivalent to an annulus on which R^p is an irrational rotation.*

From Sullivan's theorem it follows that the only wandering points of a rational map belong to preperiodic components of the Fatou set. The assumption that R

is a rational map is essential : indeed one can construct examples of entire maps with wandering domains [Ba, He4].

Whereas the proof of Sullivan's no wandering domains theorem goes beyond the purpose of this short introduction (see [Su2], [CG], [Be]) the classification of the periodic components of $F(R)$, essentially due to Fatou, is much more elementary.

Proof of the classification of periodic components of the Fatou set. It is not restrictive to assume that $R(U) = U$. Since the Julia set contains at least three points U is a hyperbolic Riemann surface and we can apply Theorem A4.11 of Appendix 4. If U contains an attracting fixed point for R then it is an attracting or a superattracting basin. The case of an irrational rotation of a disk or an annulus gives a Siegel disk or a Herman ring (U cannot be a punctured disk since the Julia set has no isolated points). The case of a map of finite order cannot occur since the degree of R is at least 2. To conclude one must show that case (ii) in Theorem A1.11 gives rise to a parabolic basin. To this purpose we will use the following lemma due to Fatou (see [CG], pp. 77-78 or [F], Section 54) :

Lemma 6.3 (Fatou's snail lemma) *Suppose U is hyperbolic, $f : U \to U$ is analytic on U and across ∂U and $f^n(z_0) \to \partial U$ for some $z_0 \in U$. Then there is a fixed point $w \in \partial U$ for f such that $f^n(z) \to w$ for all $z \in U$. Either w is an attracting fixed point or it is a parabolic fixed point with multiplier $f'(w) = 1$.*

End of the proof of Theorem 6.1 It is enough to observe that since there are no attracting periodic points in the Julia set Fatou's snail lemma produces a parabolic fixed point of R and U is a parabolic basin. □

Exercise 6.4 Show that $J(R)$ belongs to the chain–recurrent set $C(R)$. Show that all components of $F(R)$ belong to $C(R)$ except for preperiodic components which contain an eventually periodic attracting point.

As we read in [MSS, p. 197] "Even when $J(R)$ is contaminated by critical points one may think of $J(R)$ as the *hyperbolic* part of the dynamics. (The word "hyperbolic" and "elliptic" are meant here to suggest chaotic and rigid structure respectively in the dynamics). The Siegel disks and Herman rings are in the *elliptic* part of the dynamics. The attractive basins and the parabolic basins are the *properly discontinuous* part of the dynamics.
The existence of rotation domains has been for a long time an open problem (in

1919 Gaston Julia even claimed, in an incorrect paper, that Siegel disks did not exist).

Definition 6.5 *An irrational rotation $z \mapsto R_\lambda(z) = \lambda z$, $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, is called* diophantine *if α is a diophantine number. Otherwise it is called* Liouville *(since α is a Liouville number).*

In 1942 C.L. Siegel [S] showed that all holomorphic perturbations of a diophantine rotation on $\mathbb{D}$ give rise to a Siegel disk. Later V.I. Arnol'd [Ar1] showed that all diffeomorphisms of the circle which are close to a diophantine rotation are analytically conjugate to a rotation. The first examples of rational maps giving rise to diophantine rotations on annuli are due to M. Herman [He4]. These domains are foliated by quasiperiodic orbits and we will study them more closely in the next section.

6.2 Quasiperiodic orbits : Siegel disks

Let R_λ denote the map $R_\lambda(z) = \lambda z$. We recall here the notion of linearizable map already encountered in Lecture 1.

Definition 6.6 *A germ f of holomorphic diffeomorphism of $(\mathbb{C}, 0)$ with linear part λ is* linearizable *if there exists $h_f \in z\mathbb{C}\{z\}$ tangent to the identity (a linearization of f) such that $h_f^{-1} f h_f = R_\lambda$, i.e. f is locally conjugate to (its linear part) R_λ.*

Exercise 6.7 Let $R \in \mathrm{End}\,(\overline{\mathbb{C}})$ and assume that $R(0) = 0$. If R is linearizable and $R'(0) = \lambda$, $|\lambda| \leq 1$ then 0 is a stable fixed point. If $|\lambda| > 1$ then 0 is not stable.

One of the remarkable features of the holomorphic case, which carries over also to higher dimensions, is that stability of a fixed point (a topological notion) is equivalent to linearizability (an analytical notion).

Theorem 6.8 *Let $R(0) = 0$, $|R'(0)| \leq 1$. 0 is stable if and only if R is linearizable.*

Proof. The statement is non–trivial only if $\lambda = R'(0)$ has unit modulus. If R is linearizable then the linearization h_R maps a small disk $\mathbb{D}_r$ around zero conformally onto a connected open neighborhood U of 0. On U the dynamics is the same of R_λ thus 0 is stable.

Conversely assume now that 0 is stable and let $K_0 = \bigcap_{n\geq 0} R^{-n}(\mathbb{D})$. Let U_0 denote the connected component of the interior of K_0 which contains 0. Since 0 is stable $U_0 \neq \emptyset$ and one can easily see that it must also be simply connected (otherwise, if it had a hole V, surrounding it with some closed curve γ contained in U_f, since the family of iterates R^n is bounded by 1 on γ, the maximum principle leads to the same conclusion for all the points in V thus $V \subset U_0$). Applying the Riemann mapping theorem to U_0 one sees that by conjugation with the Riemann map R induces a univalent map S of the disk into itself with the same linear part λ. By Schwarz' Lemma one must have $S(z) = \lambda z$ thus R is analytically linearizable. □

When $\lambda = R'(0)$ has modulus one, is not a root of unity and 0 is stable then U_0 is conformally equivalent to a disk. The *Siegel disk* of R at 0 is the maximal connected open set containing 0 on which R is conjugated to R_λ. It contains U_0. The main result of Siegel is the following.

Theorem 6.9 (Siegel) *If an analytic map has an irrationally indifferent fixed point with the multiplier $\lambda = e^{2\pi i\alpha}$ and α is a diophantine number then the fixed point is stable and is surrounded by a disk foliated by quasiperiodic orbits analytically conjugated to the rotation by α on the circle.*

Yoccoz later showed that a weaker condition, introduced by Brjuno in 1971 and which was known to be sufficient for the existence of a Siegel disk, is indeed necessary (see Appendix 5 for the definition of Brjuno number : we refer to [Yo2], [PM] and [Mar2] for further information and to [Yo3] for the proof).

Theorem 6.10 *If α is not a Brjuno number then there exists an analytic map with an irrationally indifferent unstable fixed point with multiplier $\lambda = e^{2\pi i\alpha}$.*

For the proof of Siegel theorem under the Brjuno condition see [Br] (and [Mar2] for a pedagogical presentation). It is not known if the counterexamples of Yoccoz can occur in the category of rational maps. Indeed one has the following conjecture

Conjecture 6.11 (Douady) *If a rational map has a Siegel disk then the multiplier is $\lambda = e^{2\pi i\alpha}$ with α a Brjuno number.*

6.3 Quasiperiodic orbits : Herman rings

The existence of annuli foliated by quasiperiodic orbits analytically conjugated to a diophantine rotation on the circle is due to Herman [He1, He4].

Exercise 6.12 Show that polynomials do not have Herman rings. [Hint : apply the maximum principle to show that each bounded stable region is simply connected].

A finite *Blaschke fraction* is a rational map of the form

$$z \mapsto S(z) = \lambda \prod_{i=1}^{d} \frac{z + a_i}{1 + \overline{a_i} z} \ , \tag{6.1}$$

where $|\lambda| = 1$ and $|a_i| \neq 1$. We further assume that $a_j \neq 1/\overline{a_i}$ so that the fraction is irreducible, but $a_i = \infty$ is allowed.

Exercise 6.13 Show that S maps $\mathbb{S}^1$ into itself and $S|_{\mathbb{S}^1}$ has degree $d_0 - d_1$ where d_0 (resp. d_1) is the number of indices i such that $|a_i| < 1$ (resp. $|a_i| > 1$). Of course the degree of R is $d = d_0 + d_1$.

If one chooses $d_0 = d_1 + 1 \geq 2$ then S induces a diffeomorphism of the circle. In this case one can define the *rotation number* of S : $\alpha = \lim_{n\to\infty} \frac{\arg S^n(z)}{2\pi n}$ where $|z| = 1$. The existence of this limit and its independence on the choice of $z \in \mathbb{S}^1$ are proved in Appendix 6. One can also choose $\lambda \in \mathbb{S}^1$ in (6.1) such that $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. In this case one has the following fundamental result (due to Herman [He1] and Yoccoz [Yo1, Yo6]) :

Theorem 6.14 *All analytic diffeomorphisms of the circle with diophantine rotation number α are analytically conjugate to the rotation $z \mapsto e^{2\pi i\alpha} z$.*

Yoccoz has indeed determined the optimal arithmetical condition for the analytic conjugacy of analytic diffeomorphisms of the circle to a rotation. In the local case (i.e. if the given diffeomorphism is $\mathcal{C}^0$–close to a rotation) it is given by the same Brjuno condition which is optimal in the problem of Siegel disks. In the global case (as in Theorem 6.13 above) the optimal condition in stronger than Brjuno's condition but weaker than diophantine.

6.4 Hyperbolic rational maps

We give here a short introduction to the conjecture of the density of hyperbolicity for rational endomorphisms of the sphere.

Definition 6.15 *A rational map R is* hyperbolic *if its Julia set $J(R)$ is hyperbolic w.r.t. some conformal metric ds_R on a neighborhood of $J(R)$.*

Note that R is hyperbolic if and only if there exist $C > 0$ and $K > 1$ such that $\|(R^n)'(z)\| > CK^n$ for all $z \in J(R)$ and $n \geq 1$ (i.e. R is *expanding*).

Exercise 6.16 Show that being hyperbolic does not depend on the choice of the conformal metric. [Hint : if $ds = \omega(z)|dz|$, then $\|R'(z)\| = \frac{\omega(R(z))}{\omega(z)}|R'(z)|$].

As it is natural, in order to show that a rational map is hyperbolic, one must have a good control on the orbits of critical points : the following characterizations of hyperbolicity are easy to check (see [CG], p. 90 or [Mc1], pp. 45–46) :

Theorem 6.17 *The following conditions are equivalent :*
(i) R is hyperbolic ;
(ii) all critical points of R tend to attracting cycles under iteration ;
(iii) The postcritical set and the Julia set are disjoint : $P(R) \cap J(R) = \emptyset$.

The most important open problem in the field is

Conjecture 6.18 (Density of Hyperbolicity) *Hyperbolic maps are open and dense among all rational maps.*

Exercise 6.19 Prove that hyperbolicity is an open condition (thus the hard part in Conjecture 6.18 is density).

Whereas the conjecture of density of hyperbolicity is still open one knows that the set of structurally stable[2] rational maps is open and dense [MSS]. Thus to settle Conjecture 6.18 it suffices to prove that a structurally stable rational map is hyperbolic.

[2] In this context $R \in \mathrm{End}\,(\overline{\mathbb{C}})$ is structurally stable if it is J–stable : all endomorphisms S in a neighborhood of R are topologically conjugate to R on the Julia set, i.e. there exists a homeomorphism $h : J(R) \to J(S)$ such that $hR|_{J(R)} = S|_{J(S)}h$.

6.5 No invariant line fields conjecture

According to McMullen "Theoretical support [for Conjecture 6.18] is provided by a more fundamental conjecture, which has its roots in the quasiconformal deformation theory of rational maps and relates to Mostow rigidity of hyperbolic 3–manifolds" [Mc2, p. 5]. Let us consider once again Lattès example. In this case the Julia set (which is the whole Riemann sphere) *carries an invariant line field* : the map $z \mapsto \alpha z$, with $\alpha \in \mathbb{N}$, $\alpha \geq 2$ preserves the family of horizontal lines in $\mathbb{C}$, thus the endomorphism F preserves the images of such lines on the torus $X = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$. The quotient line family is a foliation by parallel simple closed geodesics on X. Thus R preserves the image of this foliation on the sphere $\overline{\mathbb{C}}$. Of course the image foliation must have singularities : indeed it has singular points at the four critical points of $\wp$.

$\mathbb{S}^2 = J(\text{Lattès})$ and the invariant line field.

Definition 6.20 *An* invariant line field *of a rational map R on a measurable set $E \subset \overline{\mathbb{C}}$ is the choice of a one–dimensional real subspace L_z in $T_z\overline{\mathbb{C}}$ for all $z \in E$ such that :*

(i) E has positive area ;
(ii) E is completely invariant : $R^{-1}(E) = E$;
(iii) the slope of L_z varies measurably with respect to z for all $z \in E$;
(iv) R' maps L_z into $L_{R(z)}$.

Exercise 6.21 After reading Appendix 4, show that a line field is the same

as a measurable Beltrami differential $\mu = \mu(z)\frac{d\overline{z}}{dz}$ with $|\mu| = 1$ and $f^*\mu = \mu$. [Hint : given a tangent vector $v = a(z)\frac{\partial}{\partial z}$ the line field is obtained imposing $\mu(v) = \mu(z)\frac{\overline{a(z)}}{a(z)} = 1$; conversely the real line through $a(z)\frac{\partial}{\partial z}$ corresponds to $\mu = \frac{a(z)}{\overline{a(z)}}\frac{d\overline{z}}{dz}$.]

Definition 6.22 *A rational map R admits an invariant line field on its Julia set if possess an invariant line field on a measurable subset E of its Julia set.*

Remark 6.23 The Julia set must have positive measure before it can carry an invariant line field. It is not known if a Julia set can have positive measure without being the whole Riemann sphere.

In [Mc1] the following conjecture is proposed :

Conjecture 6.24 (No invariant line fields) *A rational map R carries no invariant line field on its Julia set unless it is covered by an integral torus endomorphism.*

The main result of [McS] is the following

Theorem 6.25 *No invariant line fields implies density of hyperbolicity (Conjecture 6.24 implies Conjecture 6.18).*

Of course Lattès maps are not hyperbolic. One can show that this conjecture implies the density of hyperbolic rational maps of a fixed degree in $\mathrm{End}_d(\overline{\mathbb{C}})$.

6.6 Limit sets of Kleinian groups

As E. Ghys writes in [CGSY, p. 8] "Dès le début des années 1980, Sullivan propose un «dictionnaire» entre plusieurs théories : les groupes kleinéens, l'itération des fractions rationnelles, les équations différentielles algébriques et, plus généralement, les feuilletages transversalement holomorphes. Le point commun est un pseudogroupe de transformations holomorphes. Pour une fraction rationelle, trasformation non inversible de la sphère de Riemann, il faut penser au pseudogroupe engendré par toutes les branches de l'«inverse». Pour les groupes kleinéens, ce pseudogroupe est un groupe global : il n'y a plus de point critique et donc de ramification pour l'inverse, mais par contre, au lieu d'itérer une seule

transformation, on doit considérer les orbites d'un groupe en général très «gros». Pour les feuilletages algébriques, il s'agit du pseudogroupe d'holonomie de Reeb–Haefliger, les difficultés étant alors doubles puisque, d'un part, on a affaire à plusieurs transformations qu'il faut composer et, d'autre part, les transformations en question ne sont pas globalement définies. On trouve par example dans le dictionnaire l'analogie (enfin établie clairement) entre l'ensemble limite d'un groupe kleinéen et l'ensemble de Julia d'une fraction rationnelle. Ces analogies vont s'avérer fructueuses."

Here we will just give the statement of Sullivan's theorem proving the analogue of the no invariant line fields conjecture for Kleinian groups. The interested reader is referred to [Su1, Su3, Mc3].

A *Kleinian group* Γ is a discrete subgroup of the group $\mathrm{PSL}(2,\mathbb{C})$ of conformal automorphisms of the Riemann sphere. We assume that Γ is finitely generated, without torsion, orientation–preserving and that every abelian subgroup has infinite index.

The dynamics is obtained considering the action of Γ on the sphere $\overline{\mathbb{C}}$. This gives rise to a partition of the sphere $\overline{\mathbb{C}}$ into a non–empty open set $F(\Gamma)$ (the *regular set*) on which Γ acts freely (thus $F(\Gamma)/\Gamma$ is a finite union of hyperbolic Riemann surfaces[3]) and a closed set $J(\Gamma)$ (the *limit set*).

Definition 6.26 *The* limit set $J(\Gamma)$ *is the set of accumulation points of any orbit* $\Gamma z \subset \overline{\mathbb{C}}$.

Exercise 6.27 Show that the limit set is closed and perfect.

Exercise 6.28 Show that the limit set has the following equivalent descriptions :
(i) $J(\Gamma)$ is the smallest closed Γ–invariant set with more than two elements ;
(ii) $J(\Gamma)$ is the closure of the repelling fixed points of $\gamma \in \Gamma$;
(iii) $J(\Gamma)$ is the set of points in $\overline{\mathbb{C}}$ near which Γ does not form a normal family.

Theorem 6.29 (Sullivan) *The limit set carries no invariant line field.*

Exercise 6.30 Show that the limit set of the group generated by $z \mapsto z+1$, $z \mapsto z+\tau$ and $z \mapsto 2z$, with $\tau \in \mathbb{C} \setminus \mathbb{R}$ is $\overline{\mathbb{C}}$ and has an invariant line field. [Hint :

[3] This is the statement of Ahlfors' finiteness theorem which is the analogue – and ancestor – in this context of Sullivan's no wandering domains theorem for rational endomorphisms.

Of course this is just the universal covering of Lattès example and what goes wrong here is that the group is not discrete.]

Lecture 7. Renormalization and quasiperiodic orbits : Yoccoz's theorem

The idea of renormalization has been extremely successful in dynamics. Its origin is in statistical physics and quantum field theory.

In physics two approaches to renormalization coexist : one (and the oldest) is perturbative the other is non–perturbative. The latter grew up from the study of critical phenomena (ferromagnetism, superfluidity, polymers, conductivity of random media, etc.) and statistical physics. Here one observes that very different systems are surprisingly similar in a quantitative way, since they have the same critical exponents scaling laws (universality). At the critical transition these systems are somehow dominated by large distance correlations which are not sensitive to the details of the microscopic interactions.

COMMENT ON RENORMALIZATION IN DYNAMICAL SYSTEMS

TRIVIAL AND NONTRIVIAL FIXED POINTS

The basic idea of (non–perturbative) renormalization in studying quasiperiodic orbits is as follows : given our dynamical system $f \in \mathrm{End},(X)$ if it has some region X_0 of phase space filled by quasiperiodic orbits then its iterates form an equicontinuous family on it and $f^{n_j+1} \approx f$ for some sequence $n_j \to \infty$. Thus one can look for a region $U_0 \subset X_0$ which returns (at least approximately) to itself under n_1 iterations of f. If we then construct the quotient $X_1 = U_0/f$ identifying $x \in U_0$ with $f(x)$ the renormalized map $\mathcal{R}_{n_1} f \in \mathrm{End},(X_1)$ will be induced by the first return map to U_0 (under the iteration of f). Then f will have quasiperiodic orbits of frequency $\alpha_0 \in (0,1)$ if and only if $\mathcal{R}_{n_1} f$ has quasiperiodic orbits of frequency α_1 with $\alpha_0^{-1} = \alpha_1 + n_1$.

The process can be iterated. If one can control the sequence $\mathcal{R}_{n_j} \cdots \mathcal{R}_{n_1} f \in \mathrm{End},(X_j)$ (and the geometric limit $X_0 \to X_1 \to X_2 \to \ldots$) proving its convergence to some fixed point $f_\infty \in \mathrm{End},(X_\infty)$, where $X_\infty = \lim X_j$ then one has proved that the dynamics of f on X_0 is the same as the dynamics of f_∞ on X_∞. In the quasiperiodic case f_∞ typically is a linear automorphism of X_∞ (trivial fixed point) but other possibilities are conceivable.

In what follows we will very briefly and approximately describe Yoccoz's analysis of the renormalization of quasiperiodic orbits in the simplest case of Siegel domains.

7.1 One–step (qualitative) renormalization about an indifferent fixed point : Douady–Ghys' Theorem

In this section we begin the study of renormalization : a *qualitative* analysis of the basic construction (first return map and geometric quotient) already gives some non–trivial information on the dynamics of germs of holomorphic diffeomorphisms with an indifferent fixed point.

Let $\mathcal{Y}$ denote the set of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that all holomorphic germs of diffeomorphisms of $(\mathbb{C}, 0)$ with linear part $e^{2\pi i\alpha}$ are linearizable. Then we already know that $\mathcal{Y}$ has full measure (Siegel's Theorem) but that its complement in $\mathbb{R}\setminus\mathbb{Q}$ is a G_δ–dense. Further information on the structure of $\mathcal{Y}$ is provided by the following

Theorem 7.1 (Douady–Ghys) *$\mathcal{Y}$ is $SL(2,\mathbb{Z})$–invariant.*

We recall that $\mathrm{SL}(2,\mathbb{Z})$ is the group of matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer coefficients a, b, c, d such that $ad-bc = 1$. It acts on $\mathbb{R}\cup\{\infty\}$ as usual : $g\cdot\alpha = \frac{a\alpha+b}{c\alpha+d}$. $\mathrm{SL}(2,\mathbb{Z})$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $T\cdot\alpha = \alpha + 1$, and $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $U\cdot\alpha = -1/\alpha$.

Proof. (sketch). $\mathcal{Y}$ is clearly invariant under T, thus we only need to show that if $\alpha \in \mathcal{Y}$ then also $U\cdot\alpha = -1/\alpha \in \mathcal{Y}$.

Let $f(z) = e^{2\pi i\alpha}z + \mathrm{O}(z^2)$ and consider a domain V' bounded by

1) a segment l joining 0 to $z_0 \in \mathbb{D}^*$, $l \subset \mathbb{D}$;
2) its image $f(l)$;
3) a curve l' joining z_0 to $f(z_0)$.

We choose l' and z_0 (sufficiently close to 0) so that l, l' and $f(l)$ do not intersect except at their extremities. Note that l and $f(l)$ form an angle of $2\pi\alpha$ at 0. Then glueing l to $f(l)$ one obtains a topological manifold $\overline{V}$ with boundary which is homeomorphic to $\overline{\mathbb{D}}$. With the induced complex structure its interior is biholomorphic to $\mathbb{D}$. Let us now consider the *first return map* $g_{V'}$ to the domain V' (this is well defined if z is choosen with $|z|$ small enough) : if $z \in V'$ (and $|z|$ is small enough) we define $g_{V'}(z) = f^n(z)$ where n (depends on z) is defined asking that $f(z), \ldots, f^{n-1}(z) \notin V'$ and $f^n(z) \in V'$, i.e. $n = \inf\{k \in \mathbb{N},\, k \geq 1,\, f^k(z) \in V'\}$. Then it is easy to check that $n = \left[\frac{1}{\alpha}\right]$ or $n = \left[\frac{1}{\alpha}\right] + 1$. The first return map $g_{V'}$ induces a map $g_{\overline{V}}$ on a neighborhood of $0 \in \overline{V}$ and finally a germ g of holomorphic

diffeomorphism at $0 \in \mathbb{D}$ ($gp = pg_{\overline{V}}$, where p is the projection from $\overline{V}$ to the disk $\mathbb{D}$). It is easy to check that $g(z) = e^{-2\pi i/\alpha} z + \mathrm{O}(z^2)$ (note that in the passage from V' to $\mathbb{D}$ through $\overline{V}$ the angle $2\pi\alpha$ at the origin is mapped in 2π).

To each orbit of f near 0 corresponds an orbit of g near 0. In particular

- f is linearizable if and only if g is linearizable ;
- if f has a periodic orbit near 0 then also g has a periodic orbit ;
- if f has a point of instability (i.e. a point whose iterates leave a neighborhood of 0) then also g has a point of instability (which will escape even more rapidly).

In particular these statements show that $\alpha \in \mathcal{Y}$ if and only if $-1/\alpha \in \mathcal{Y}$. $\square$

7.2 Distorsion estimates and quantitive one–step renormalization

Let $f(z) = e^{2\pi i\alpha} z + \mathrm{O}(z^2)$ be analytic and univalent on $\mathbb{D}$ and let $E : \mathbb{H} \to \mathbb{D}^*$ be the exponential map $E(z) = e^{2\pi i z}$. Then f lifts to a map $F : \mathbb{H} \to \mathbb{C}$ such that

$$E \circ F = f \circ E \,, \tag{7.1}$$

$$F : \mathbb{H} \to \mathbb{C} \text{ is univalent} \,, \tag{7.2}$$

$$F(z) = z + \alpha + \varphi(z) \text{ where } \varphi \text{ is } \mathbb{Z}-\text{periodic and } \lim_{\Im m\, z \to +\infty} \varphi(z) = 0 \,. \tag{7.3}$$

We will denote $S(\alpha)$ the space of univalent functions F verifying (7.1), (7.2) and (7.3).

Exercise 7.2 Show that $S(\alpha)$ is compact and that it is the universal cover of $S_{e^{2\pi i\alpha}} = \{f : \mathbb{D} \to \mathbb{C}\,,\ \text{univalent}\,,\ f(0) = 0\,,\ f'(0) = e^{2\pi i\alpha}\}$.

Exercise 7.3 Use the result of the previous exercise to show that if φ is as in (7.3) and $z \in \mathbb{H}$ then

$$|\varphi'(z)| \leq \frac{2\exp(-2\pi\, \Im m\, z)}{1 - \exp(-2\pi\, \Im m\, z)} \,, \tag{7.4}$$

$$|\varphi(z)| \leq -\frac{1}{\pi} \log(1 - \exp(-2\pi\, \Im m\, z)) \,. \tag{7.5}$$

[Hint : From (7.1) one gets $\varphi'(z) = E(z)\frac{f'(E(z))}{f(E(z))} - 1$ thus (7.4) can be deduced from the standard distortion estimate $\left| w\frac{f'(w)}{f(w)} - 1 \right| \leq \frac{2|w|}{1-|w|}$ for univalent functions on the unit disk, see [CG] or [Po]. Then (7.5) can be deduced by integration.]

Let $r > 0$, $\mathbb{H}_r = \mathbb{H} + ir$. It is clear that if $F \in S(\alpha)$ and r is sufficiently large then F is very close to the translation $z \mapsto z + \alpha$ for $z \in \mathbb{H}_r$. Indeed using the compactness of $S(\alpha)$ and Exercise 7.3 one can prove the following :

Proposition 7.4 *Let $\alpha \neq 0$. There exists a universal constant $c_0 > 0$ (i.e. independent of α) such that for all $F \in S(\alpha)$ and for all $z \in \mathbb{H}_{t(\alpha)}$ where*

$$t(\alpha) = \frac{1}{2\pi} \log \alpha^{-1} + c_0 \ , \tag{7.6}$$

one has

$$|F(z) - z - \alpha| \leq \frac{\alpha}{4} \ . \tag{7.7}$$

Proof. Let $\varphi(z) = F(z) - z - \alpha$. Then φ is defined on $\mathbb{H}$, $\mathbb{Z}$–periodic and vanishes as $\Im m\, z \to +\infty$ thus by Exercise 7.3 one gets

$$|\varphi(z)| \leq -\frac{1}{\pi} \log(1 - \exp(-2\pi\, \Im m\, z)) \leq -\frac{1}{\pi} \log(1 - \exp(-2_0))$$

if $\Im m\, z > t(\alpha)$. Taking c_0 large enough we get (7.7). □

Given F, the lowest admissible value $t(F, \alpha)$ of $t(\alpha)$ such that (7.7) holds for all $z \in \mathbb{H}_{t(F,\alpha)}$ represents the height in the upper half plane $\mathbb{H}$ at which the strong nonlinearities of F manifest themselves. When $\Im m\, z > t(F, \alpha)$, F is very close to the translation $T_\alpha(z) = z + \alpha$. This is equivalent to say that f is very close to the rotation by $2\pi\alpha$ when $z \in \mathbb{D}$ is sufficiently small.

An example of strong nonlinearity is of course a fixed point : if $F(z) = z + \alpha + \frac{1}{2\pi i} e^{2\pi i z}$, $\alpha > 0$, then $z = -\frac{1}{4} + \frac{i}{2\pi} \log(2\pi\alpha)^{-1}$ is fixed and $t(F, \alpha) \geq \frac{1}{2\pi} \log(2\pi\alpha)^{-1}$.

As in the proof of Douady–Ghys theorem we can now construct the first return map in the "strip" B delimited by $l = [it(\alpha), +i\infty[$, $F(l)$ and the segment $[it(\alpha), F(it(\alpha))]$. Given z in B we can iterate F until $\Re e\, F^n(z) > 1$. If $\Im m\, z \geq t(\alpha) + c$ for some $c > 0$ then $z' = F^n(z) - 1 \in B$ and $z \mapsto z'$ is the first return map in the strip B. Glueing l and $F(l)$ by F one obtains a Riemann surface S corresponding to $\operatorname{int} B$ and biholomorphic to $\mathbb{D}^*$. This induces a map $g \in S_{e^{2\pi i/\alpha}}$ which lifts to $G \in S(\alpha^{-1})$.

As we can see we have three steps :

(g) *glueing* l and $F(l)$ following F we get a topological manifold with boundary whose interior is biholomorphic to the standard half–cylinder ;

(u) *uniformization* of the manifold obtaining a standard cylinder ;

(d) *developing* the standard cylinder on the plane.

The biholomorphism $H = dug$ which "glues, uniformizes and develops" the "strip" B into the strip of width 1 conjugates F with the translation by 1

$$H(F(z)) = H(z) + 1$$

and the renormalized map is

$$G = \mathcal{R}_{a_1} F = H F^{a_1} T_{-1} H^{-1} ,$$

where a_1 is the integer part of α^{-1}. It is immediate to check that if $\Im m\, z$ is large then $H(z) = \frac{z}{\alpha} + \dots$ and one can then show the following (see [Yo2], pp. 32–33)

Proposition 7.5 *Let $\alpha \in (0,1)$, $F \in S(\alpha)$ and $t(\alpha) > 0$ such that if $\Im m\, z \geq t(\alpha)$ then $|F(z) - z - \alpha| \leq \alpha/4$. There exists $G \in S(\alpha^{-1})$ such that if $z \in \mathbb{H}$, $\Im m\, z \geq t(\alpha)$ and $F^i(z) \in \mathbb{H}$ for all $i = 0, 1, \dots, n-1$ but $F^n(z) \notin \mathbb{H}$ then there exists $z' \in \mathbb{C}$ such that*

1. *$\Im m\, z' \geq \alpha^{-1}(\Im m\, z - t(\alpha) - c_1)$, where $c_1 > 0$ is a universal constant ;*
2. *There exists an integer m such that $0 \leq m < n$ and $G^m(z') \notin \mathbb{H}$.*

7.3 Convergence of the renormalization scheme and linearization : Yoccoz's theorem

The estimates (7.6) and (7.7) are the fundamental ingredient of Yoccoz's proof of the existence of Siegel disks for all germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$ with linear part $e^{2\pi i\alpha}$ where α satisfies an arithmetical condition, weaker than the diophantine condition, known as *Brjuno condition*. Moreover Yoccoz proves that this condition is also necessary : if α is not a Brjuno number then one can construct a germ which has a sequence of periodic orbits which accumulate the origin, thus it cannot be linearizable.

of the lower bound (5.2) together with Proposition 2.15 and the following elementary properties of the conformal capacity.

Exercise 5.15 Let $U \subset C$ be a simply connected open set, $U \neq \mathbb{C}$, and let $z_0 \in U$. Let d be the distance of z_0 from the complement of U in $\mathbb{C}$ and let $C(U, z_0)$ denote the conformal capacity of U w.r.t. z_0. Then

$$d \leq C(U, z_0) \leq 4d \, . \tag{5.21}$$

The main result of Yoccoz can be very simply stated as

$$\mathcal{Y} = \{\alpha \in \mathbb{R} \setminus \mathbb{Q} \mid \ B(\alpha) < +\infty\} = \text{Brjuno numbers} ,$$

but he proves much more than the above :

Theorem 7.2
(a) If $B(\alpha) = +\infty$ there exists a non–linearizable germ $f \in S_{e^{2\pi i\alpha}}$;
(b) If $B(\alpha) < +\infty$ then $r(\alpha) > 0$ and

$$|\log r(\alpha) + B(\alpha)| \leq C , \tag{7.1}$$

where C is a universal constant (i.e. independent of α) ;
(c) For all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all Brjuno numbers α one has

$$-B(\alpha) - C \leq \log r(P_{e^{2\pi i\alpha}}) \leq -(1-\varepsilon)B(\alpha) + C_\varepsilon \tag{5.12}$$

where C is a universal constant (i.e. independent of α and ε.

The remarkable consequence of (5.11) and (5.12) is that the Brjuno function not only identifies the set $\mathcal{Y}$ but also gives a rather precise estimate of the size of the Siegel disks.

The proof of the Theorem of Yoccoz (5.9 above) uses a method, invented by Yoccoz himself, known as *geometric renormalization*. Roughly speaking it is a quantitative version of the topological construction of Douady–Ghys described in Chapter 4 and which shows that the set $\mathcal{Y}$ is $\mathrm{SL}(2,\mathbb{Z})$–invariant. Whereas the construction of non–linearizable germs $f \in G_{e^{2\pi i\alpha}}$ when $B(\alpha) = +\infty$ and Yoccoz's upper bound

$$\log r(\alpha) \leq C - B(\alpha)$$

go far beyond the scope of these lectures, it is not too difficult to give an idea of how Yoccoz proves Theorem 5.1, i.e. the lower bound

$$\log r(\alpha) \geq -C - B(\alpha) .$$

From this Proposition one can conclude the proof as follows. Let us recall that $K_f = \cap_{n\geq 0} f^{-n}(\mathbb{D})$ is the maximal compact f–invariant set containing 0. Let

$F \in S(\alpha)$ be the lift of $f \in S_{e^{2\pi i\alpha}}$ and let $K_F \subset \mathbb{C}$ be defined as the cover of K_f : $K_F = E^{-1}(K_f)$. It is immediate to check that

$$d_F = \sup\{\Im m\, z \mid z \in \mathbb{C} \setminus K_F\} = -\frac{1}{2\pi} \log \operatorname{dist}(0, \mathbb{C} \setminus K_f) \ .$$

Thus by (5.21) one gets

$$\exp(-2\pi d_F) \leq C(K_f, 0) \leq 4 \exp(-2\pi d_F) \ .$$

Theorem 5.1 is therefore equivalent to the lower bound

$$\sup_{F \in S(\alpha)} d_F \leq \frac{1}{2\pi} B(\alpha) + C \tag{5.22}$$

for some universal constant $C > 0$.

Assume that (5.22) is not true and that there exist $\alpha \in \mathbb{R} \setminus \mathbb{Q} \cap (0, 1/2)$ with $B(\alpha) < +\infty$, $F \in S(\alpha)$, $z \in \mathbb{H}$ and $n > 0$ such that

$$\begin{aligned} \Im m\, F^n(z) &\leq 0 \ , \\ \Im m\, z &\geq \frac{1}{2\pi} B(\alpha) + C \ . \end{aligned}$$

Let us choose α, F and z so that n is as small as possible. By Proposition 5.16, if $C > c_0$, one gets

$$\begin{aligned} \Im m\, z' &\geq \alpha^{-1}[\Im m\, z - t(\alpha) - c_1] \\ &\geq \alpha^{-1} \left[\frac{1}{2\pi} (B(\alpha) - \log \alpha^{-1}) + C - c_0 - c_1 \right] \ . \end{aligned}$$

By the functional equation of B one gets

$$\Im m\, z' \geq \frac{1}{2\pi} B(\alpha^{-1}) + \alpha^{-1}[C - c_0 - c_1] \geq \frac{1}{2\pi} B(\alpha^{-1}) + C$$

provided that $C \geq 2(c_0 + c_1)$. But Proposition 5.16 shows that this contradicts the minimality of n and we must therefore conclude that (5.22) holds. □

A nice description of the proof of the upper bound $\log r(\alpha) \leq C - B(\alpha)$ can be found in the Bourbaki seminar of Ricardo Perez–Marco [PM1].

7.2 Quadratic Polynomials. Mandelbrot set

The simplest non trivial rational maps are *quadratic polynomials*. Since every quadratic polynomial is globally conjugated on $\mathbb{C}$ by an affine map to exactly one polynomial of the family

$$Q_c(z) = z^2 + c \ , \quad c \in \mathbb{C}$$

the parameter space is exactly $\mathbb{C}$.

In the case of quadratic polynomials Q_c the only finite critical point is $z = 0$ (the other is the superattracting fixed point at infinity).

The *filled–in Julia set* of Q_c is the set

$$K_c = \{ z \in \mathbb{C} \, , \, \sup_{n \geq 0} |Q_c^n(z)| < +\infty \} \ .$$

The Julia set of Q_c is the boundary of K_c.

Exercise 5.23 Show that if $R = (1 + \sqrt{1 + 4|c|})/2$ and $|z| > R$ then $|Q_c(z)| > |z|$ thus $K_c = \cap_{n \geq 0} Q_c^{-n}(\{|z| \leq R\})$. Thus K_c is compact. Moreover show that $\mathbb{C} \setminus K_c$ is connected.

In order to study K_c a very useful tool is its potential G_c (or Green function). Since Q_c behaves at ∞ as z^2 one can define for large enough $|z|$

$$G_c(z) = \lim_{n \to +\infty} 2^{-n} \log^+ |Q_c^n(z)| \ ,$$

and also the *Bóttcher function*

$$g_c(z) = \lim_{n \to +\infty} (Q_c^n(z))^{2^{-n}} \ .$$

Exercise 5.20 Show that

(i) $G_c : \mathbb{C} \to \mathbb{R}^+$ is continuous, harmonic outside K_c and $G_c(z) = \log |z| + \mathrm{O}(|z|^{-2})$ as $|z| \to \infty$;
(ii) $K_c = \{ z \in \mathbb{C} \mid G_c(z) = 0 \}$;
(iii) $G_c(Q_c(z)) = 2 G_c(z)$;
(iv) $g_c(Q_c(z)) = (g_c(z))^2$;
(v) $\log |g_c(z)| = G_c(z)$ and $g_c(z) = z + \mathrm{O}(|z|^{-1})$ as $|z| \to \infty$;
(vi) g_c maps conformally $\{G_c(z) > G_c(0)\}$ onto $\{|z| > \exp(G_c(0))\}$.

Theorem 5.21 *K_c is connected if and only if $0 \in K_c$.*

Proof. If $0 \in K_c$ then the Green function has no critical point in $\mathbb{C} \setminus K_c$ and its level curves are all simple. Thus K_c must be connected.

Conversely assume that $0 \in \mathbb{C} \setminus K_c$. The critical points of G_c are 0 and all its preimages. The largest critical value of G_c is exactly $G_c(0) > 0$ and the corresponding level curve is a figure eight. Thus K_c is not connected. □

As a subproduct of the previous proof we see that if K_c is connected then g_c maps conformally the exterior of K_c onto the outer unit disk $\mathbb{C} \setminus \overline{\mathbb{D}}$. We will denote $h_c = g_c^{-1}$.

Definition 5.22 *The* Mandelbrot set *is the set M of parameter values c for which K_c is connected.*

Exercise 5.23 Show that M is compact, $\mathbb{C} \setminus M$ is connected, M is symmetric w.r.t. the real axis and $M \cap \mathbb{R} = [-2, 1/4]$.

7.3 Renormalization and quadratic–like maps

In the first part of this lecture we have seen how renormalization can be used to study the stability of irrationally indifferent fixed points. Here we want to illustrate how it is used to study the Mandelbrot set.

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Let $Q_c(z) = z^2 + c$ with $c \in M$. An iterate Q_c^n is *renormalizable* if there exist two disks U and V containing the origin, with $\overline{U} \subset V$, such that (a) $Q_c^n : U \to V$ is a proper map of degree two and (b) $Q_c^{nk}(0) \in U$ for all $k > 0$. This means that although Q_c^n is a polynomial of degree 2^n it behaves like a polynomial of degree two on a suitable neighborhood of the critical point. The restriction $Q_c^n|_U$ is called a *quadratic–like* map. The fundamental result of Douady and Hubbard says that any quadratic like map is topologically conjugate to a quadratic polynomial $Q_{c'}$ and condition (b) implies that $c' \in M$ and, with suitable normalization, is unique.

The concept of renormalization explains much of the self–similarity in the Mandelbrot set and in the bifurcation diagram for real quadratic polynomials.

A quadratic polynomial Q_c is *infinitely renormalizable* if Q_c^n is renormalizable for infinitely many $n > 1$.

The most important results in the field are :

Theorem (Yoccoz) *Let $c \in M$. Then either Q_c is inifinitely renormalizable or $J(Q_c)$ has no invariant line field and M is locally connected at c.*

For a skectch of the proof see [Mc ?]

Theorem (McMullen) *If Q_c is an infinitely renormalizable real quadratic polynomial then $J(Q_c)$ has no invariant line field*

The two previous results show that *every component of the Mandelbrot set which meets the real axis is hyperbolic.*

Lecture 8. Hamiltonian systems

In this Lecture we will very briefly recall some well–known facts on symplectic manifolds and Hamiltonian systems. Very good references are [AKN] and [AbM].

8.1 Symplectic Manifolds and Hamiltonian Systems

Definition 8.1 *A* $\mathcal{C}^\infty$ symplectic manifold *is a* $2l$*–dimensional* $\mathcal{C}^\infty$ *manifold* M *equipped with a non–degenerate* $\mathcal{C}^\infty$ *two–form (the symplectic form)* ω*. A* $\mathcal{C}^\infty$ *map* $f : U \to M'$ *where* $U \subset M$ *is open and* M' *is also symplectic (with symplectic form* ω'*) is* symplectic *(or* canonical*) if* $f^*\omega' = \omega$*.*

The simplest (but important) examples of symplectic manifolds are :

- $M = \mathbb{R}^{2l} \ni (p_1, \ldots, p_l, q_1, \ldots, q_l)$, $\omega = \sum_{i=1}^{l} dp_i \wedge dq_i$ (*standard symplectic structure*). If U and V are two open sets in $\mathbb{R}^{2l}$ and $f : U \to V$ then f is symplectic if and only if its Jacobian matrix $J_f \in \mathrm{Sp}\,(l, \mathbb{R})$, the Lie group of $2l \times 2l$ real matrices A such that $A^T \mathcal{I} A = \mathcal{I}$, where $\mathcal{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- $M = T^*N$ where N is a $\mathcal{C}^\infty$ Riemannian manifold. This is the typical situation in classical mechanics. If $(q_1, \ldots, q_l)$ are local coordinates in N and $(p_1, \ldots, p_l)$ are the corresponding local coordinates in the cotangent space at a point, then $\omega = \sum_{i=1}^{l} dp_i \wedge dq_i$.
- $M = \mathbb{T}^{2l}$, $\omega = \sum_{i=1}^{l} d\theta_i \wedge d\theta_{i+l}$.

Theorem 8.2 (Darboux) *Each symplectic manifold* M *has an atlas* $(U_\alpha, \varphi_\alpha)_{\alpha \in \mathcal{A}}$ *such that on* $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^{2l}$ *one has* $\omega = \varphi_\alpha^* \sum_{i=1}^{l} dp_i \wedge dq_i$ *(the standard symplectic structure on* $\mathbb{R}^{2l}$*). The transition maps* $\varphi_\alpha \circ \varphi_\beta^{-1}$ *are symplectic diffeomorphisms, i.e. their Jacobians* $J_{\alpha\beta}(x) \in Sp\,(l, \mathbb{R})$ *for all* $x \in \varphi_\beta(U_\alpha \cap U_\beta)$*.*

The atlas given by Darboux's Theorem and the corresponding local coordinates are called symplectic.

Definition 8.3 *A* Hamiltonian function *on a symplectic manifold* (M, ω) *is a function* $H \in \mathcal{C}^\infty(M, \mathbb{R})$*. The* Hamiltonian vector field *associated to* H *is the unique* $X_H \in \mathcal{C}^\infty(M, TM)$ *such that* $i_{X_H}\omega = dH$*.*

Note that in symplectic local coordinates a Hamiltonian vector field takes the form

$$X_H = \sum_{i=1}^{l} -\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} \ , \tag{8.1}$$

and the associated ordinary differential equations are the classical Hamilton's equations of the motion of a conservative mechanical system with l degrees of freedom :

$$\dot{p_i} = -\frac{\partial H}{\partial q_i} \ , \quad \dot{q_i} = \frac{\partial H}{\partial p_i} \ , \quad 1 \le i \le l \ . \tag{8.2}$$

Clearly the Hamiltonian function is a first integral of (8.2). The coordinates q_i are also called "generalized coordinates" and the p_i their "conjugate momenta". In many problems arising from celestial mechanics the flow is not complete due to the unavoidable occurance of collisions, but we will always assume completeness of the Hamiltonian flow.

Definition 8.4 *The* Poisson bracket *of two functions $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$ defined on an open subset of (M, ω) is $\{F, G\} := X_G F = \omega(X_F, X_G) = -X_F G$, thus $X_{\{F,G\}} = -[X_F, X_G]$. Two functions F, G are* in involution *if $\{F, G\} = 0$, i. e. when their hamiltonian flows commute.*

Exercise 8.5 Show that the *Hamiltonian flow* $\Phi : \mathbb{R} \times M \to M$ is symplectic : for all $t \in \mathbb{R}$ one has $\Phi(t, \cdot)^* \omega = \omega$. [Hint : use Cartan's formula $\frac{d}{dt}|_{t=0} \Phi(t, \cdot)^* \omega = d(i_{X_H}\omega) + i_{X_H} d\omega$, where $X_H = \frac{d}{dt}|_{t=0}\Phi(t, \cdot)$ is the Hamiltonian vector field associated to Φ.]

The importance of Exercise 8.5 is that to make symplectic coordinate changes of a Hamiltonian vector field it is sufficient to change the varables in the corresponding Hamiltonian function. This is a simpler operation, both conceptually and computationally.

8.2 Integrable Systems

An especially interesting example of symplectic manifold is $M = \mathbb{R}^l \times \mathbb{T}^l$ which can be identified with the cotangent bundle of the l–dimensional torus $\mathbb{T}^l = \mathbb{R}^l/(2\pi\mathbb{Z})^l$. This manifold has a natural symplectic structure defined by the closed 2–form $\omega = \sum_{i=1}^{l} dJ_i \wedge d\vartheta_i$ where $(J_1, \dots J_l, \vartheta_1, \dots \vartheta_l)$ are coordinates on $\mathbb{R}^l \times \mathbb{T}^l$.

Definition 8.6 *Let U denote an open connected subset of $\mathbb{R}^l$. Whenever an Hamiltonian system can be reduced by a symplectic change of coordinates to a function $H : U \times \mathbb{T}^l \to \mathbb{R}$ which* does not depend on the angular variables ϑ *one says that the system is* completely canonically integrable *and the variables J are called* action variables.

Note that in this case Hamilton's equations (8.2) take the particularly simple form

$$\dot{J}_i = -\frac{\partial H}{\partial \vartheta_i} = 0 \ , \quad \dot{\vartheta}_i = \frac{\partial H}{\partial J_i} \ , \quad i = 1, \dots, l$$

and the flow leaves invariant the l–dimensional torus $J =$ constant. The motion is therefore bounded and quasiperiodic (or periodic).

Being completely canonically integrable is a stronger requirement than integrability by quadratures or complete integrability (see [AKN] for their discussion). In the latter case one requires the existence of l independent first integrals in involution but their joint level–set may well be non compact (this is already the case in the two body problem for non negative energy values) and the flow does not need to be quasiperiodic (scattering states).

The main risult in the theory of completely canonically integrable systems is the celebrated

Theorem 8.7 (Arnol'd–Liouville) *Let $H \in \mathcal{C}^\infty(M, \mathbb{R})$ and assume that $F_1, \dots, F_l \in \mathcal{C}^\infty(M, \mathbb{R})$ are l first integrals in involution for the Hamiltonian flow associated to H. Let $a \in \mathbb{R}^l$ be such that $M_a = \{m \in M \mid F_i(m) = a_i \, \forall i = 1, \dots, l\}$ is not empty and assume that the l functions $F_1, \dots, F_l$ are independent*[1] *in a neighborhood of M_a. Then if M_a is compact and connected it is diffeomorphic to the l–torus. Moreover there exists an invariant open subset V of M which contains M_a and is symplectically diffeomorphic to $U \times \mathbb{T}^l$, where U is an open subset of $\mathbb{R}^l$.*

Arnol'd–Liouville's Theorem thus assures that the existence of sufficiently many first integrals together with the compactness and connectedness of their level set guarantees complete canonical integrability.

8.3 Examples of completely canonically integrable systems

[1] As usual $F_1, \dots, F_l$ are independent if $dF_1 \wedge \dots \wedge dF_l \neq 0$.

In this section we will briefly describe some examples of completely canonical integrable systems.

Example 8.8 : Harmonic oscillators. Let $M = \mathbb{R}^{2l}$ with the standard symplectic structure, $S \in \mathrm{GL}(2l, \mathbb{R})$ be symmetric and positive definite. Consider the Hamiltonian system $H(x) = \frac{1}{2} x^T S x$. This is completely integrable. Indeed if J is a symplectic matrix which diagonalizes S, in the variables $y = J^{-1}x$ the Hamiltonian will be

$$H(y) = \sum_{i=1}^{2l} \frac{\lambda_i y_i^2 + \lambda_{i+l} y_{i+l}^2}{2} \ ,$$

where $\lambda_i \, , \, i = 1, \ldots, 2l$ are the eigenvalues of S. Then the functions $F_i = \frac{\lambda_i y_i^2 + \lambda_{i+l} y_{i+l}^2}{2}$, $i = 1, \ldots, l$, are independent first integrals in involution and their common level set is compact and connected (since $\lambda_i > 0$ for all i). The symplectic transformation to action–angle variables is

$$y_i = \sqrt{2J_i \sqrt{\lambda_{i+l}/\lambda_i}} \cos \chi_i \ , \quad y_{i+l} = \sqrt{2J_i \sqrt{\lambda_i/\lambda_{i+l}}} \sin \chi_i \ , \quad i = 1, \ldots, l \ .$$

Example 8.9 : The two body problem. The Hamiltonian $\mathcal{H} : T^*(\mathbb{R}^3 \setminus \{0\}) \mapsto \mathbb{R}$ of the two-body problem in the center of mass frame is (we have assumed $G = 1$, where G is the universal gravitational constant)

$$\mathcal{H}(p, q) = \frac{1}{2\mu} \|p\|^2 - \frac{m_0 m}{\|q\|}$$

where $\mu = m_0 m/(m_0 + m)$ is the reduced mass of the system.

It is well-known that for negative energy the solutions are ellipses with one focus at the origin (i. e. the center of mass). These are called *keplerian orbits*. The shape and the position of the ellipse in space are determined from the knowledge of the major semiaxis a, the eccentricity e, the angle of inclination i of its plane w.r.t. the horizontal plane $q_3 = 0$, the argument of perihelion ω and the longitude of the ascending node Ω. The position of the planet along the ellipse is determined by the mean anomaly l, which is proportional to the area swept by the position vector q of the planet starting from the perihelion.

The systems admits 5 *independent* first integrals : the total energy $\mathcal{H}$, the three components of the angular momentum $q \wedge p$ and one of the components of the Laplace–Runge–Lenz vector $A = p \wedge q \wedge p - \frac{m_0 m q}{\|q\|}$. Among these integrals one can choose three integrals in involution and construct the completely canonical

transformation to action–angle variables. The other two integrals are responsible for the proper complete degeneration of the Kepler problem : one can choose action–angle variables so that the Hamiltonian depends only on one of the actions. Indeed the Delaunay action–angle variables $(L, G, \Theta, l, g, \theta)$ are related to the orbital elements as follows :

$$L = \mu\sqrt{(m_0+m)a}\ ,\ \ G = L\sqrt{1-e^2}\ ,\ \ \Theta = G\cos i\ ,\ \ l\ ,\ \ g = \omega\ ,\ \ \theta = \Omega\ .$$

Note that G is the modulus of angular momentum $q \wedge p$, thus Θ is its projection along the q_3–axis. One has the obvious limitation $|\Theta| \leq G$. The new Hamiltonian reads $\mathcal{H} = -\frac{\mu^3(m_0+m)^2}{2L^2}$.

The relation among Delaunay variables and the original momentum–position (p, q) variables is much more subtle and will not be discussed here.

8.4 Quasi–integrable Hamiltonian Systems

The importance of completely canonically integrable Hamiltonian systems is due both to the fact that their flows can be studied in great detail and that many problems in mathematical physics can be considered as *perturbations* of integrable systems. The most famous example is given by the motion of the planets in the Solar System (see [Ma2] and references therein for an introduction). If the (weak) mutual attraction between the planets is neglected the system decouples into several independent Kepler problems and it is completely integrable. Exactly this problem gave origin in the 18th century to "perturbation theory" whose modern formulation is mainly due to the monumental work of Henri Poincaré [P]. The goal of pertubation theory is to understand the dynamics of a "perturbed" system which is close to a well–understood one (usually an integrable system).

Following Poincaré [P], the *fundamental problem of dynamics* is the study of quasi–integrable Hamiltonian systems : let $\varepsilon_0 > 0$,

Definition 8.10 *A* quasi–integrable *Hamiltonian system is a function* $\mathcal{H} \in \mathcal{C}^\infty((-\varepsilon_0, \varepsilon_0) \times M, \mathbb{R})$ *such that the Hamiltonian function* $H = \mathcal{H}(0, \cdot) : M \to \mathbb{R}$ *is completely canonically integrable.*

Using the canonical transformation to action–angle variables associated to $\mathcal{H}(0, \cdot)$, Hamiltonians $\mathcal{H} : (-\varepsilon_0, \varepsilon_0) \times U \times \mathbb{T}^l \mapsto \mathbb{R}$ (smooth or analytic) of the form

$$\mathcal{H}(\varepsilon, J, \chi) = h_0(J) + \varepsilon f(J, \chi)\ , \tag{8.3}$$

where $f \in \mathcal{C}^\infty(U \times \mathbb{T}^l, \mathbb{R})$, are typical examples of quasi–integrable Hamiltonian systems.

The most ambitious program would be to prove that quasi–integrable Hamiltonian systems are indeed integrable : i.e. to show that there exists a one–parameter family V_ε of open connected invariant subsets of M which are symplectically diffeomorphic to $U_\varepsilon \times \mathbb{T}^l$ where $U_\varepsilon \subset \mathbb{R}^l$ is open, connected and such that if $(\tilde{J}, \tilde{\chi})$ are the coordinates in $U_\varepsilon \times \mathbb{T}^l$ one has $\mathcal{H}(\varepsilon, \cdot, \cdot) \mid_{V_\varepsilon} = h_\varepsilon(\tilde{J})$ for some smooth one–parameter family of smooth function $h_\varepsilon : U_\varepsilon \times \mathbb{T}^l \to \mathbb{R}$.

In general this is asking too much : a result of Poincaré shows that in general quasi–integrable Hamiltonian systems are not completely integrable (in addition to [P], Tome I, Chapitre V, see [BFGG] for a nice discussion of the consequences of this problem and a related result of Fermi).

Theorem 8.11 (Poincaré) *Consider a quasi–integrable Hamiltonian of the form (8.1), $l \geq 2$. Assume that the two following genericity assumptions are satisfied : (1)* non–degeneracy *: $\det\left(\frac{\partial^2 h_0}{\partial J_i \partial J_k}\right) \neq 0$ on U ; (2)* generic perturbations *: for all $J \in U$ and for all $k \in \mathbb{Z}^l \setminus \{0\}$ either the k–th Fourier coefficient $\hat{f}_k(J)$ of f does not vanish or there exists $k' \in \mathbb{Z}^l \setminus \{0\}$ parallel to k such that $\hat{f}_{k'}(J) \neq 0$. Then the system is not a smooth one–parameter family of completely canonically integrable Hamiltonians.*

One can also recall the following theorem of Markus and Meyer [MM]

Theorem 8.12 *Generically hamiltonian systems are neither completely canonically integrable nor ergodic*

Exercise 8.13 Prove Poincaré's Theorem following these lines. Using the notations introduced above, if the system were completely canonically integrable then the new actions $\tilde{J}$ would be a system of l independent first integrals of the Hamiltonian flow of $\mathcal{H}$ in involution. Writing them explicitly in terms of the old local coordinates (J, χ) one has

$$\tilde{J} = J + \varepsilon \tilde{J}_1(J, \chi) + \mathcal{O}(\varepsilon^2) \ , \tag{8.4}$$

for some smooth function $\tilde{J}_1 : V_\varepsilon \to U_\varepsilon$. Imposing that $\{\tilde{J}, \mathcal{H}\} = \mathcal{O}(\varepsilon^2)$ leads to

the system of linear partial differential equations

$$\sum_{i=1}^{l} \frac{\partial h_0}{\partial J_i} \frac{\partial \tilde{J}_{1j}}{\partial \chi_i} = \frac{\partial f}{\partial \chi_j} \ , \quad j = 1, \ldots, l. \tag{8.5}$$

Using Fourier series try to find a smooth solution to these equations . . ..

8.5 KAM Theory, Nekhoroshev Theorem, Arnol'd Diffusion

Despite Theorem 8.11, most results on quasi–integrable systems have been obtained under the assumption of non–degeneracy (i.e. the hessian matrix of h_0 is non degenerate thus the frequency map $J \mapsto \nu_0(J) = \frac{\partial h_0}{\partial J}(J) \in \mathbb{R}^l$ is a local diffeomorphism) but accepting the fact that one cannot hope for integrability on open sets.

The general picture is provided by KAM [Ar1,Ga, Bo, Yo1] and Nekhoroshev [Ne, Lo] theorems : if ε is sufficiently small, most initial conditions (w.r.t. Lebesgue measure) lie on invariant l–dimensional lagrangian tori carrying quasiperiodic motions with Diophantine frequencies. The action variables corresponding to these orbits will remain ϵ–close to their initial values for all times. The complement of this set is open and dense and it is connected if $l \geq 3$. It contains a connected ($l \geq 3$) web $\mathcal{R}$ of resonant zones corresponding to $\mathbb{Z}^l$–linearly dependent frequencies : $\cup_{k \in \mathbb{Z}^l} \{J \in U \, , \nu_0(J) \cdot k = 0\} \times \mathbb{T}^l$. Motion along these resonances cannot be excluded (see [Ar2] for an explicit example), resulting in a variation of $\mathcal{O}(1)$ of the actions in a finite time[2]. But if the hamiltonian is *analytic* and h_0 is *steep* (for example convex or quasi convex) then this variation is very slow : it takes a time at least $\mathcal{O}\left(\exp\left(\frac{1}{\varepsilon^a}\right)\right)$ to change the actions of $\mathcal{O}(\varepsilon^b)$, where a and b are two positive constants. Moreover each invariant torus has a neighborhood filled in with trajectiories which remain close to it for an even longer time. Indeed, if h_0 is quasi–convex one can prove [GM] that all trajectories starting at a distance of order $\rho < \rho^*$ from a Diophantine l–torus of exponent τ will remain close to it for a time $\mathcal{O}\left(\exp\left(\exp\left(\frac{\rho^*}{\rho}\right)^{1/\tau+1}\right)\right)$.

[2] It is conjectured [AKN, p. 189] that generically quasi–integrable hamiltonians with more than two degrees of freedom are topologically unstable

One of the consequences of KAM theorem [Pö] is the existence, for sufficiently small values of ε, of a *Cantor set* N_ε of values of the frequencies ν for which the Hamiltonian system (8.3) has smooth invariant tori with linear flow. Moreover there exists a homeomorphism $F_\varepsilon : N_\varepsilon \times \mathbb{T}^l \to U \times \mathbb{T}^l$ ε–close to the identity, Whitney smooth w.r.t. the first factor and analytic w.r.t. the second (if the Hamiltonian (8.3) is analytic) which transforms Hamilton's equations into $\dot{\nu} = 0$, $\dot{\varphi} = \nu$. This foliation into invariant tori is thus parametrized over a Cantor set and hence nowhere dense. It exhibits the phenomenon of "anisotropic differentiability" since it is much more regular tangentially to these tori than transversally to them (see also [BHS]).

Lecture 9. Twist maps and the standard map

In this lecture we will briefly discuss some basic facts concerning the dynamics of twist maps and of the standard map.

9.1 Twist maps

We let $\mathbb{S}^1$ denote the circle $\mathbb{R}/\mathbb{Z}$ and $\theta(\mathrm{mod}\,1)$ denote the standard parameter of $\mathbb{S}^1$ and x denote the corresponding parameter of its universal cover $\mathbb{R}$. We will let $y \in \mathbb{R}$ denote the standard parameter of the second factor of $\mathbb{S}^1 \times \mathbb{R}$ so that (θ, y) is a global system of coordinates for the cylinder $\mathbb{S}^1 \times \mathbb{R}$ and (x, y) is a global system of coordinates of its universal cover $\mathbb{R}^2$.

We will denote with U those open subsets of the cylinder which intersect each vertical line $\{\theta\} \times \mathbb{R}$ in an open interval (which may be infinie on one or both ends). We will always consider maps f which are diffeomorphisms from U onto an open subset $f(U) \subset \mathbb{S}^1 \times \mathbb{R}$ which also intersects each vertical line in an open interval and which are *orientation preserving* and *area preserving*. Since we are in two dimensions the condition that f be area preserving is the same as requiring that f be *symplectic*, i.e. that f preserves the usual symplectic two–form $\omega = d\theta \wedge dy$ (or, if $\tilde{f}$ denotes the lift of f to the universal cover so that $\tilde{f}(x+1, y) = \tilde{f}(x, y) + (1, 0)$, $\tilde{f}$ preserves $dx \wedge dy$).

Definition 9.1 *An orientation preserving symplectic homeomorphism f is* exact *if for all Jordan curve $\gamma \subset U$ which is homotopically non–trivial and with zero area the area above C and below γ is the same as the area above C and below $f(\gamma)$, where $C = \mathbb{S}^1 \times \{y\}$ and y is chosen such that both γ and $f(\gamma)$ lie above C.*

Exercise 9.2 If f is a $\mathcal{C}^1$ diffeomorphism then being exact is equivalent to

$$\int_0^1 y(\gamma(s)) \frac{d}{ds} x(\gamma(s)) ds = \int_0^1 y(f(\gamma(s)) \frac{d}{ds} x(f(\gamma(s))) ds \,, \tag{9.1}$$

where $\gamma : \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}$ is any embedding of a homotopically non–trivial curve.

Definition 9.3 *An orientation preserving exact symplectic homeomorphism f is* a monotone twist *map if for each $\theta_0 \in \mathbb{S}^1$ the mapping $\theta \circ f : U \cap \{\theta_0\} \times \mathbb{R} \to \mathbb{S}^1$ is a local homeomorphism (i.e. monotone decreasing* – negative *monotone twist* – *or monotone increasing* – positive *monotone twist).*

Exercise 9.4 If f is a $\mathcal{C}^1$ diffeomorphism then being positive (resp. negative) monotone twist is equivalent to require that there exists $\varepsilon > 0$ such that for all $z \in \mathbb{S}^1 \times \mathbb{R}$ one has

$$\frac{\partial \theta \circ f(z)}{\partial y} > \varepsilon \ \text{ (resp. } < -\varepsilon) \ . \tag{9.2}$$

Geometrically this condition states that the image of a segment x =constant under $\tilde{f}$ forms a graph over the x axis.

Example 9.5 Integrable twist maps : $f(x, y) = (x + r(y), y)$ where r is monotone.

Example 9.6 A plane billiard table is a two dimensional bounded domain M with a piecewise smooth boundary. The billiard dynamical system is generated by the free motion of a mass–point (the billiard ball) subject to the elastic reflection in the boundary. Consider the manifold U of unit tangent vectors (x, v) with the inward direction v and the footpoint x on the boundary ∂M. If the boundary consists of one component then V is an annulus $\mathbb{T}^1 \times I$ and the billiard transformation maps (x, v) to (x_1, v_1) obtained considering the intersection with ∂M of the straight line through x in the direction of v. At x_1 the vector v reflects in ∂M to a new vector v_1. Parametrize ∂M by arc length t and let α be the angle between v and the direction of the boundary at the point $x(t)$, so as to use (t, α) as coordinates for U, $\alpha \in [0, \pi]$. Show that the billiard transformation preserves the symplectic two form $\sin\alpha d\alpha \wedge dt$.

Example 9.7 TIME–ONE MAPS OF PERIODIC HAMILTONIANS OF THE FORM $H(x, y, t) = y^2/2 + V(x, y, t)$. THE CONVERSE IS ALSO TRUE !(Moser's Theorem : Monotone twist mappings and the calculus of variations, Ergodic Theorem and Dynamical Systems **6** 401–413 (1986))

9.2 The standard map. Variational principles

Frenkel–Kantorova model of solid state physics : let us take an arbitrary locally finite configuration $\{x_n\}$ of points on $\mathbb{R}^1$ and write down the formal expression for their potential energy in the form

$$E(\{x_n\}) = \frac{1}{2}\sum (x_n - x_{n-1} - \beta)^2 + \frac{k}{4\pi^2}\sum (1 - \cos 2\pi x_n) \ . \tag{9.3}$$

Here local finiteness means that each interval contains a finite number of points, β and k are parameters. The first sum is the energy of internal interaction of points and the second sum is their energy in an external periodic field.

The main problem is to find the configurations of minimal energy or *ground states*. The first sum is minimized choosing $x_n - x_{n-1} = \beta$ whereas the second is minimized choosing $x_n \in \mathbb{Z}$. The competition between these two sums depending on values of β and k determines the complicated structure of the set of ground states.

A *ground state* is a configuration $\{x_n^{(0)}\}$ such that for all configurations $\{x_n\}$ which differs from $\{x_n^{(0)}\}$ by the position of finitely many points (thus $x_n = x_n^{(0)}$ for all $|n| > N$) the difference $E(\{x_n\}) - E(\{x_n^{(0)}\}) \geq 0$. Clearly a necessary condition is

$$\left(\frac{\partial E(\{x_n\})}{\partial x_n}\right)_{x_n = x_n^{(0)}} = 0 \ , \quad -\infty < n < +\infty \ . \tag{9.4}$$

More explicitly (9.4) means that

$$x_{n+1}^{(0)} - 2x_n^{(0)} + x_{n-1}^{(0)} = \frac{k}{2\pi} \sin 2\pi x_n^{(0)} \ , \quad -\infty < n < +\infty \ . \tag{9.5}$$

If one sets $y_n = x_n - x_{n-1}$ then (9.5) can be rewritten as the following exact area preserving twist map

$$y_{n+1} = y_n + \frac{k}{2\pi} \sin 2\pi x_n \ , \quad x_{n+1} = x_n + y_{n+1} \ , \tag{9.6}$$

known as *standard map* or *Chirikov–Taylor map*. The literature on the standard map is huge !

Exercise 9.8 Show that all maps of the form $f_k(x, y) = (x + y + ku(x), y + ku(x))$ where u is $\mathbb{Z}$-periodic and $\int_0^1 u(x)dx = 0$ are exact symplectic monotone twist maps.

9.3 Birkhoff's theory and non–existence of invariant curves. Some open problems

Let A be an annulus, $\partial A = \gamma_+ \cup \gamma_-$, f map A into itself preserving the boundary circles $\gamma_\pm$.

Definition 9.9 *The annulus A is* stable *if the boundary circles γ_+ and γ_- have neighborhoods such that no orbit starting in one neighborhood reaches the other. If the annulus A is not stable it is called a* Birkhoff region of instability *or a* ring of instability.

Clearly if f has a homotopically non–trivial invariant curve inside A then A is stable. The converse is also true :

Theorem 9.10 (Birkhoff) *If A is a monotone exact twist map and A is a stable annulus then there exists a homotopically non–trivial invariant curve $\gamma \subset A$. Moreover γ is the graph of a Lipschitz function.*

Remark 9.11 J. Mather has proved that in a ring of instability there exists a point z whose orbit is α–asymptotic to γ_- and ω–asymptotic to γ_+.

The classical results of Denjoy on diffeomorphisms of the circle show that given an invariant curve γ, if the rotation number α of $f|_\gamma$ is irrational then :

- if $f|_\gamma \in \mathcal{C}^2$ then every orbit is dense in γ ;
- there exists examples of $f|_\gamma \in \mathcal{C}^{2-\varepsilon}$, $\varepsilon > 0$, such that no orbit is dense in γ and the limit set L of the orbits of the points of γ is a Cantor set (Denjoy minimal set).

Since the twist maps we are interested in are smooth one sees that limit sets different from γ may appear if and only if the invariant curve γ loses smoothness.

Inside a Birkhoff region one also has many periodic points.

Theorem 9.12 (Poincaré's last geometric theorem, due to Birkhoff) *Let f be an exact symplectic monotone twist map which preserves boundary components $\gamma_\pm$ of an annulus A. Let $\alpha_\pm$ denote the rotation numbers of $f|_{\gamma_\pm}$ and let $p/q \in \mathbb{Q}$ such that $\alpha_- \leq p/q \leq \alpha_+$. Then f has at least two distinct orbits which are p/q–periodic (i.e. such that $\tilde{f}^q(x,y) = (x,y) + (p,0)$.*

For the standard map KAM theory assures the existence of many invariant circles when k is small enough. However if k is too large there are no invariant circles any more as first proved by J. Mather :

Theorem 9.13 *If $k > 4/3$ the standard map has no homotopically non–trivial invariant circle.*

Proof. Let

$$h(x,x') = \frac{1}{2}(x-x')^2 + u(x)$$

where $u(x) = -\frac{k}{4\pi^2}\cos 2\pi x$. Then $y = -h_x$, $y' = h_{x'}$. Let γ be an invariant circle, $\phi : \mathbb{S}^1 \to \mathbb{R}$ denote the Lipschitz function of which γ is the graph, g denote the bi–Lipschitz orientation preserving homeomoprhism of $\mathbb{S}^1$ induced by f :

$$f(\theta, \phi(\theta)) = (g(\theta), \phi(g(\theta)))\,.$$

Since

$$0 = -\partial_x(h_x(x, g(x)) + h_{x'}(g^{-1}(x), x)) = \frac{dg^{-1}}{dx} + \frac{dg}{dx} - u'' - 2$$

if one defines

$$L = \max\left(\sup\frac{|g(x) - g(x')|}{|x - x'|}, \sup\frac{|g^{-1}(x) - g^{-1}(x')|}{|x - x'|}\right)$$

and

$$m = \min u''\,, \quad M = \max u''$$

one gets

$$0 < 2L^{-1} < 2 + m\,, \quad L + L^{-1} \leq 2 + M$$

thus

$$\frac{2+m}{2} + \frac{2}{2+m} \leq 2 + M$$

which implies $|k| < 4/3$. □

Lecture 10. A review of Ergodic theory. Kolmogorov–Sinai entropy

The recommended readings for this lecture are [You] and [Mn].

10.1 Review of abstract ergodic theory

Ergodic theory is an attempt to study the statistical behaviour of orbits of dynamical systems restricting the attention to their asymptotic distribution. One waits until all transients have been wiped off and looks for an invariant probability measure describing the distribution of typical orbits.

This approach is especially fruitful for systems with a very strong sensitivity to initial conditions.

Let $(X, \mathcal{A}, \mu)$ be a probability space, i.e. X is a set, $\mathcal{A}$ is a σ–algebra of subsets of X and μ is a measure on $(X, \mathcal{A})$ such that $\mu(X) = 1$.

Definition 10.1 *A map $S : X \to X$ is* measurable *if for all $A \in \mathcal{A}$, $S^{-1}(A) \in \mathcal{A}$. A measurable transformation is* non singular *if $\mu(S^{-1}(A)) = 0$ for all $A \in \mathcal{A}$ such that $\mu(A) = 0$. A non singular measurable transformation is* measure–preserving *if for all $A \in \mathcal{A}$, $\mu(S^{-1}(A)) = \mu(A)$.*

In the above definition $S^{-1}(A) = \{x \in X \mid S(x) \in A\}$ and S need not be invertible. If S is invertible with a non singular measurable inverse and it preserves the measure μ then $\mu(S^{-1}(A)) = \mu(A) = \mu(S(A))$, $\forall\ A \in \mathcal{A}$.

Exercise 10.2 Why should one use S^{-1} instead of S in Definition 10.1 ?

Definition 10.3 *A* measurable dynamical system $(X, \mathcal{A}, \mu, S)$ *is the datum of a probability space $(X, \mathcal{A}, \mu)$ and of a measure preserving map $S : X \to X$.*

Exercise 10.4 Prove Poincaré's recurrence theorem : for all $A \in \mathcal{A}$ the subset A_0 of the points $x \in A$ such that $S^n(x) \in A$ for infinitely many $n \in \mathbb{N}$ belongs to $\mathcal{A}$ and $\mu(A) = \mu(A_0)$. Let Xbe a separable metric space with metric d, let $\mathcal{A}$ be its borelian σ-algebra, μ a probability measure and $S : X \to X$ a measure preserving

map. To each point $x \in X$ we associate its ω-*limit* set

$$\omega(x) := \{y \in X \mid \liminf_{n\to\infty} d(S^n(x), y) = 0\}.$$

Use Poincaré's recurrence theorem to show that $\mu(\{x \in X \mid x \notin \omega(x)\}) = 0$.

An especially important case is given by X Riemannian manifold with metric tensor g, $\mathcal{A}$ the σ–algebra of borelians, and S preserves a measure μ_ρ absolutely continuous w.r.t. the volume on S with density ρ.

Exercise 10.5 Show that in this case S preserves μ_ρ if and only if $\int_{S^{-1}(A)} \rho(x)\, d\mathrm{Vol}_g(x) = \int_A \rho(x)\, d\mathrm{Vol}_g(x)$ for all $A \in \mathcal{A}$. ∎

A very important problem in ergodic theory is to determine the invariant measure.

Exercise 10.6 Let $X = [0,1]$, $S : X \to X$ non singular. Assume that S is piecewise monotonic and $\mathcal{C}^1$, i.e. there exists a finite or countable decomposition of the interval $[0,1]$ in subintervals $[a_i, a_{i+1}]$, $i \in \mathcal{I}$, on which S si monotonic and of class $\mathcal{C}^1$ in their interior. On each of these subintervals an inverse branch S_i^{-1} of S is well–defined. Show that a measure $\mu_\rho(x) = \rho(x)dx$ is S–invariant if and only if $\rho(x) = \sum_{i\in\mathcal{I}_x} \frac{\rho(S_i^{-1}(x))}{|S'(S_i^{-1}(x))|}$, where $\mathcal{I}_x$ denotes the subset of $\mathcal{I}$ corresponding to the indices i such that $S_i^{-1}(x) \neq \emptyset$. Use this result to check that the following probability measures are invariant :

(i) Ulam–Von Neumann's map $S(x) = 4x(1-x)$, $\rho(x)dx = \frac{dx}{\pi\sqrt{x(1-x)}}$;

(ii) p–adic map $S(x) = px (\mathrm{mod}\, 1)$, $\rho(x)dx = dx$;

(iii) Gauss' map $\frac{S(x)=\{1}{x\}}$ if $x \neq 0$, $S(0) = 0$, where $\{x\}$ denotes the fractional part of x, $\rho(x)dx = \frac{dx}{(1+x)\ \log\ 2}$.

Exercise 10.7 Let X be a compact metric space, $S : X \to X$ a continuous endomorphism of X, $\mathcal{B}$ the σ-algebra of borelian subsets of X. Prove that there exists at least an S–invariant probability measure on X. [Hint : Let $\mathcal{M}(X)$ denote the topological space of all probability measures on X with the usual weak–$*$ topology and consider the continuous map $S^* : \mathcal{M}(X) \to \mathcal{M}(X)$ defined by $(S^*\mu)(A) = \mu(S^{-1}(A))$ for all $A \in \mathcal{B}$. A probability measure is invariant if $S^*\mu = \mu$. Take any measure $\mu_0 \in \mathcal{M}(X)$ and consider the sequence $\mu_m = \frac{1}{m}\sum_{j=0}^{m-1} S^{*m}\mu_0$ and use the compactness of $\mathcal{M}(X)$.]

The first question one asks about a measurable dynamical system is whether or not it can be decomposed into smaller non–interacting subsystems. This is

equivalent to show that the interpretation of the invariant probability measure as the frequency of visit of measurable sets by typical orbits is correct.

Given a set $A \in \mathcal{A}$ we denote χ_A its characteristic function.

Definition 10.8 *The* frequency of visit $\nu(x, A)$ *of the set* A *by the orbit of* x *is the limit (when it exists)*

$$\nu(x, A) = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n-1} \chi_A(S^j x). \qquad [10.1]$$

Exercise 10.9 Prove that for all $A \in \mathcal{A}$ and for μ–almost every point $x \in X$ the frequency of visit $\nu(x, A)$ exists. [Hint : Look at M. Keane's lectures in [BKS] for an especially nice proof of it.]

Exercise 10.10 Give examples of systems with the frequency of visit independent on the choice of the initial point x and examples with dependence.

Definition 10.11 *A measurable dynamical system* $(X, \mathcal{A}, \mu, S)$ *is* ergodic *if for all* $A \in \mathcal{A}$ *one has* $\nu(x, A) = \mu(A)$ *for* μ–a.e. $x \in X$.

Theorem 10.12 (Birkhoff Ergodic Theorem) *Let* $(X, \mathcal{A}, \mu, S)$ *be a measurable dynamical system and let* $f \in L^1(X, \mathcal{A}, \mu)$. *For* μ–a.e. $x \in X$ *the limit*

$$\hat{f}(x) := \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(S^j x)$$

exists and it is called the time average *of* f *along the orbit of* $x \in X$. *If the system is ergodic then* $\hat{f}(x) = \int_X f d\mu$ a.e.

Theorem 10.13 *Let* $(X, \mathcal{A}, \mu, S)$ *a measurable dynamical system. The following properties are equivalent :*

1) *ergodicity ;*
2) measurable indecomponibily *: for all invariant* $A \in \mathcal{A}$ *one has* $\mu(A) = 0$ *or* $\mu(A) = 1$ *;*
3) *if* $f \in L^1(X, \mathcal{A}, \mu)$ *is invariant then* f *is constant* μ–a.e. *;*
4) *if* $f \in L^1(X, \mathcal{A}, \mu)$ *then* $\int_X f d\mu = \hat{f}(x)$ *for* μ–a.e. $x \in X$ *;*
5) $\forall\ A, B \in \mathcal{A}$ *one has* $\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu(S^{-j}(A) \cap B) = \mu(A)\mu(B)$.

Definition 10.14 *A measurable dynamical system* $(X, \mathcal{A}, \mu, S)$ *is* mixing *if* $\forall\ A, B \in \mathcal{A}$ *one has*

$$\exists \lim_{n \to +\infty} \mu(S^{-n}(A) \cap B) = \mu(A)\mu(B). \qquad [10.2]$$

Clearly mixing implies ergodicity. Keeping in mind the probabilistic interpretation of the measure it means that asymptotically the two events $S^n(x) \in A$ and $x \in B$ become independent.

Exercise 10.15 Show that the p–adic map is mixing. Show that the hyperbolic automorphisms of the torus are mixing.

10.2 Metric entropy. Bernoulli schemes

Entropy measures the amount of randomness or complexity of a measurable dynamical system.

Let $\Delta^{(m)}$ denote the standard m–simplex of $\mathbb{R}^m$,

$$\Delta^{(m)} = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \in [0,1],\ \sum_{i=1}^{m} x_i = 1\}\ ,$$

and let $H : \Delta^{(m)} \to [0, +\infty]$ denote the function

$$H(p_1, \ldots, p_m) = -\sum_{i=1}^{m} p_i \log p_i\ , \qquad [10.3]$$

(with the convention $0 \log 0 = 0$).

If α and β are two partitions of $\mathcal{A}$, their *joint partition* $\alpha \vee \beta$ is $\{A \cap B\ ,\ A \in \alpha,\ B \in \beta\}$. Given n partitions $\alpha_1, \ldots, \alpha_n$ we will denote $\bigvee_{i=1}^{n} \alpha_i$ their joint partition. If S is measurable and non–singular and α is a partition, $S^{-1}\alpha$ is the partition defined by the subsets $\{S^{-1}A\ ,\ A \in \alpha\}$. Finally a partition β is *finer* than α, denoted $\alpha < \beta$, if $\forall\ B \in \beta\ \exists\ A \in \alpha$ such that $B \subset A$.

The entropy $H(\alpha)$ of a partition $\alpha = \{A_1, \ldots, A_m\}$ is given by $H(\alpha) = -\sum_{i=1}^{m} \mu(A_i) \log \mu(A_i)$.

Definition 10.16 *Let* $(X, \mathcal{A}, \mu, S)$ *be a measurable dynamical system and* α *a partition. The* entropy of S w.r.t. the partition α *is*

$$h(S, \alpha) := \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} S^{-i}\alpha\right) \qquad [10.4]$$

The entropy of S *is*

$$h(S) := \sup\{h(S,\alpha),\ \alpha \text{ isafinitepartitionof} X\}. \qquad [10.5]$$

Remark 10.17 Using the strict convexity of $x \log x$ on $\mathbb{R}_+$, one can prove the existence of the limit [10.4]. Indeed the sequence $\frac{1}{n} H\left(\bigvee_{i=0}^{n-1} S^{-i}\alpha\right)$ is non–negative monotonic non increasing. Thus $h(S,\alpha) \geq 0$ for all α.

Remark 10.18 The entropy of a partition α measures the amount of information obtained making a measurement by means of a device which distinguishes points of X with the resolution prescribed by $\{\mathcal{A}_1, \ldots, \mathcal{A}_m\} = \alpha$. If $x \in X$ and we consider the orbit of x up to time $n-1$ $x,\ Sx,\ S^2x,\ \ldots,\ S^{n-1}x$, since α is a partition of X the points $S^i x$, $0 \leq i \leq n-1$, belong to exactly one of the sets of α : $x_i \in A_{k_i}$ with $k_i \in \{1, \ldots, m\}$ for all $i = 0, \ldots, n-1$. $H\left(\vee_{i=0}^{n-1} S^{-i}\alpha\right)$ measures the information obtained from the knowledge of the distribution w.r.t. α of a segment of orbit of length n. Thus $\frac{1}{n} H\left(\bigvee_{i=0}^{n-1} S^{-i}\alpha\right)$ is the average amount of information per unit of time and $h(S,\alpha)$ is the amount of information (asymptotically) obtained at each iteration of the dynamical system from the knowledge of the distribution of the orbit of a point w.r.t. the partition α.

The previous remark is made rigorous by the following theorem [Mn1].

Theorem 10.19 (Shannon-Breiman-McMillan) *Let $(X, \mathcal{A}, \mu, S)$ be an ergodic measurable dynamical system, α a finite partition of X. Given $x \in X$ let $\alpha^n(x)$ be the element of $\bigvee_{i=0}^{n-1} S^{-i}\alpha$ which contains x. For μ–a.e. $x \in X$ one has*

$$h(S,\alpha) = \lim_{n\to\infty} -\frac{1}{n} \log \mu(\alpha^n(x)). \qquad [10.6]$$

Remark 10.20 The previous theorem admits the following interpretation : if a system is ergodic then there exists a non–negative number h such that $\forall\ \varepsilon > 0$ if α is a sufficiently fine partition of X then there exists a positive integer N such that for all $n \geq N$ there is a subset X_n of X with measure $\mu(X_n) > 1 - \varepsilon$ and made of approximately e^{nh} elements of $\bigvee_{i=0}^{n-1} S^{-i}\alpha$, each measuring about e^{-nh}.

Let X be a compact metric space and $\mathcal{A}$ be the Borel σ-algebra. Brin e Katok [M. Brin and A. Katok, Lecture Notes in Mathematics 1007 (1983) 30–38] gave a "topological version" of Shannon-Breiman-McMillan's Theorem. Let $B(x,\varepsilon)$ be

the ball of center $x \in X$ and radius ε. Let $S : X \to X$ be continuous and preserving the probability measure $\mu : \mathcal{A} \to [0,1]$. Let

$$B(x,\varepsilon,n) := \{y \in X \mid d(S^i x, S^i y) \leq \varepsilon \text{ forall } i = 0,\ldots,n-1\},$$

i.e. $B(x,\varepsilon,n)$ is the set of points $y \in X$ whose orbit stays at a distance at most ε from the orbit of x for at least $n-1$ iterations. Then one has

Theorem 10.21 (Brin-Katok) *Assume that $(X,\mathcal{A},\mu,S)$ is ergodic. For μ–a.e. $x \in X$ one has*

$$\sup_{\varepsilon>0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B(x,\varepsilon,n)) = h(S). \qquad [10.7]$$

Exercise 10.22 The entropy of an isometry is zero.

When the entropy is positive some of the observables are not predictable.

A system is *chaotic* if it has *positive entropy*. Brin-Katok's Theorem together with Poincaré recurrence theorem show that the orbits of chaotic systems are subject to two apparently contrasting requirements. On one hand almost every orbit is recurrent. On the other hand the probability that two orbits stay close to each other for an inteval of time of length n decays exponentially with n. Since two initially close orbits must come infinitely many times close to their origin, if the entropy is positive they cannot be correlated. Tipically they will separate one from the other and return at different times n.

To this complexity of the motions one associates the notion of chaos and shows how it can be impossible to compute the values that an observable will assume from the knowledge of the past.

Remark 10.23 To compute the entropy one can use the following important result of Kolmogorov and Sinai : if α is a partition of X which generates the σ-algebra $\mathcal{A}$ the entropy of $(X,\mathcal{A},\mu,S)$ is simply given by

$$h(S) = h(S,\alpha). \qquad [10.8]$$

We recall that α *generates* $\mathcal{A}$ iff $\bigvee_{-\infty}^{+\infty} S^{-i}\alpha = \mathcal{A} \bmod(0)$ if S is invertible, $\bigvee_{i=0}^{\infty} S^{-i}\alpha = \mathcal{A} \bmod(0)$ if S is not invertible.

Exercise 10.24 Show that the entropy of the p–adic map is $\log p$.

Exercise 10.25 Show that the entropy of a linear automorphism σ of the torus $\mathbb{T}^l$ is $\sum_{|\lambda_i|>1} \log|\lambda_i|$ where $\lambda_1,\ldots,\lambda_l$ are the eigenvalues of σ.

Lecture 11. Lyapunov exponents. Skew–products

The purpose of this lecture is to introduce the positive entropy conjecture for the standard map. This is one of the most important open problems in low–dimensional dynamics.

11.1 Lyapunov exponents

Lyapunov characteristic exponents measure the rate of exponential divergence of nearby trajectories in phase space. This is closely related to entropy.

First of all we need a general result which extends Birkhoff's ergodic theorem to the non–commuting case :

Theorem 11.1 (multiplicative ergodic theorem, Oseledec (1968)) *Let $(X, \mathcal{A}, \mu, S)$ be ergodic. Let $T : X \to GL(m, \mathbb{R})$ be measurable such that*

$$\int_X \log^+ \|T(x)\| \, d\mu < +\infty \qquad [11.1]$$

where $\log^+ u = \max(0, \log u)$. Let

$$T_x^n := T(S^{n-1}(x))T(S^{n-2}(x)) \cdots T(x) = \prod_{j=1}^{n} T(S^{n-j}(x)). \qquad [11.2]$$

For μ–a.e. $x \in X$ the limit

$$\lim_{n\to\infty} ((T_x^n)^T T_x^t)^{1/2n} = \Lambda_x \qquad [11.3]$$

exists (where $(T_x^n)^T$ denotes the transposed of T_x^n) and is a symmetric semi–definite positive matrix.

Definition 11.2 *The logarithms of the eigenvalues of the matrix Λ_x are called* Lyapunov characteristic exponents *of the system $(X, \mathcal{A}, \mu, S, T)$.*

In what follows we will consider the characteristic exponents ordered in a decreasing sequence $\lambda_1(x) \geq \lambda_2(x) \geq \cdots$ where each exponents appears as many times as its multiplicity. Note that ergodicity implies that they are constant μ–a.e. .

Let $\lambda^{(1)} > \lambda^{(2)} > \cdots$ denote the characteristic exponents listed without counting multiplicities $m^{(1)}, m^{(2)}, \ldots$. Let $E_x^{(i)}$ be the vector subspace of $\mathbb{R}^m$ corresponding to the eigenvalues $\leq \exp \lambda^{(i)}$ of Λ_x. One obtains a "filtration" of $\mathbb{R}^m$ into subspaces

$$\mathbb{R}^m = E_x^{(1)} \supset E_x^{(2)} \supset \cdots \quad . \tag{11.4}$$

Moreover one has the following theorem :

Theorem 11.3 *Let $(X, \mathcal{A}, \mu, S)$ be as Theorem 11.1. For μ-a.e. $x \in X$, if $v \in E_x^{(i)} \setminus E_x^{(i+1)}$, one has*

$$\exists \lim_{n\to\infty} \frac{1}{n} \log \|T_x^n v\| = \lambda^{(i)}. \tag{11.5}$$

For all $v \in \mathbb{R}^m \setminus E_x^{(2)}$ (i.e. for a.e. $v \in \mathbb{R}^m$ w.r.t. Lebsegue measure) the limit [11.5] is the largest characteristic exponent $\lambda^{(1)}$.

Remark 11.4 If $m = 1$ Oseledec's theorem reduces to Birkhoff's theorem (with the restriction that the functions f must be the logarithm of a positive measurable function. The further difficulty which Oseledec's theorem overcomes is the non–commutativity of products of matrices when $m > 1$. In the one–dimensional case, when S is a (piecewise) smooth map and $T = S'$, for μ-q.o. $x \in X$ one has

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log |T_x^n| = \lim_{n\to\infty} \frac{1}{n} \log \prod_{j=1}^{n} S'(S^{n-j}(x)) =$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log(S'(S^j(x)) = \int_X \log |S'(x)| \, d\mu$$

Exercise 11.5 Show that the Lyapunov exponent of the Ulam–Von Neumann map (Exercise 10.6) is $\log 2$. Show that the Lyapunov exponent of Gauss' map is $\frac{\pi^2}{6 \log 2}$. [Hint : Expand $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ and use $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi}{12}$.]

Let us now assume that X is $\mathbb{R}^l$ or a compact riemannian manifold, $\mathcal{A}$ the σ-algebra of borelian subsets, $S : X \to X$ a piecewise differentiable map and μ an invariant ergodic probability measure (se $X = \mathbb{R}^l$ we will assume that the support of μ is compact).

We choose

$$T(x) = \left(\frac{\partial S_i}{\partial x_j}(x) \right)_{i,j=1,\ldots,l} \in \mathrm{GL}\,(l, \mathbb{R}) \tag{11.6}$$

i.e. DS. By the chain rule one has

$$\left(\frac{\partial (S^n)_i}{\partial x_j}(x)\right) = \prod_{k=1}^{n} T(S^{n-k}(x)) = T_x^n \qquad [11.7]$$

thus an infinitesimal change $\delta x(0)$ after n iterations becomes $\delta x(n) = T_x^n \delta x(0)$ and by Theorem 11.3 for almost every choice of $\delta x(0)$ one has

$$\delta x(n) \sim e^{n\lambda^{(1)}} \delta x(0) \qquad [11.8]$$

and the (exponential) instability of the trajectories corresponds to $\lambda^{(1)} > 0$, where $\lambda^{(1)}$ is the largest Lyapunov exponents.

11.2 Entropy and Lyapunov exponents

Lyapunov exponents, in particular positive exponents, give the rates at which nearby orbits diverge. They provide us with a "geometric" way of measuring the complexity of the orbits of a dynamical system. On the other hand metric entropy is a "probabilistic" way of measuring their complexity. It measures their randomness in the sense of information theory.

These two approaches seem completely independent but already Brin–Katok's theorem suggests a possible relation. Roughly speaking one can say that entropy is created by the exponential divergence of nearby orbits. Then one must distinguish conservative systems where all the expansion goes back into the system to make entropy (hence one obtains Pesin's formula below) from non–conservative ones where some of the expansion is "wasted" to construct singular invariant measures (hence one obtains Ruelle's inequality below).

Let X be a compact riemannian manifold, $S : X \to X$ a $\mathcal{C}^2$ diffeomorphism of X, $\mathcal{A}$ the σ-algebra of borelian subsets of X and μ an invariant ergodic probability measure.

Theorem 11.6 (Ruelle's inequality) *For $\mathcal{C}^1$ maps one has*

$$h(S) \leq \sum_{\lambda^{(i)}>0} \lambda^{(i)} m_i. \qquad [11.9]$$

Theorem 11.7 (Pesin's formula) *If the invariant measure μ is equivalent to the Riemannian measure on X then*

$$h(S) = \sum_{\lambda^{(i)}>0} \lambda^{(i)} m_i. \qquad [11.10]$$

Exercise 11.8 Use Pesin's formula to compute the entropy of linear hyperbolic automorphisms of tori.

11.3 Entropy and standard map

The textbook of Sinai states two important conjectures on the behaviour of the orbits of the standard map [9.6] when the parameter $k \neq 0$ or k is large enough ([Si], p. 144).

Note that the standard map can be naturally reduced to a map of the two–dimensional torus if one considers also $y (\mathrm{mod}\, 1)$.

Positive entropy conjecture *The metric entropy of the standard map is positive for all values of $k \neq 0$ and tends to infinity as $k \to +\infty$.*

Numerical experiments agree with the lower bound $\log(|k|/2)$ for the entropy of the standard map but the above conjecture is not proved (or disproved) for any single value of k.

One of the difficulties to be overcome is the possible existence of elliptic islands : this has been show to occur for arbitrarily large values of k [Du].

Another interesting conjecture on the behaviour of the orbits of the standard map applies only to those values of k for which there are no invariant curves (by Mather's theorem 9.13 we know that this occurs if $|k| > 4/3$.

Diffusion conjecture *If $|k| > 4/3$ then there exists a positive Haar measure subset A of initial conditions (y_0, x_0) for which*

$$\int_A y_n^2 dy_0 dx_0 \approx \text{constant}\, n \ .$$

11.4 Herman's subharmonicity method

Let $r > 0$, we denote $\mathbb{D}_r^n$ the n–dimensional polydisc of radius r in $\mathbb{C}^n$, i.e. the set of $(z_1, \ldots, z_n) \in \mathbb{C}^n$ such that $|z_i| \leq r$ for all $i = 1, \ldots n$; we also denote $\mathbb{T}_r^n$ the n–dimensional polycircle of radius r (or distinguished boundary of $\mathbb{D}_r^n$), i.e. the

set of $(z_1, \ldots, z_n) \in \mathbb{C}^n$ such that $|z_i| = r$ for all $i = 1, \ldots n$. We will denote m the normalized Haar measure on $\mathbb{T}^n_r$.

We consider a map f holomorphic in a neighborhood of $0 \in \mathbb{C}^n$ which takes values also in $\mathbb{C}^n$ and such that

- there exists $r_0 > 0$ such that f is holomorphic in a neighborhood of $\mathbb{D}^n_{r_0}$;
- $f(\mathbb{D}^n_{r_0}) \subset \mathbb{D}^n_{r_0}$, $f(\mathbb{T}^n_{r_0}) \subset \mathbb{T}^n_{r_0}$;
- $f(0) = 0$;
- $f_* m = m$, where m is the normalized Haar measure on $\mathbb{T}^n_{r_0}$.

Let B be a complex Banach algebra and $\| \|$ be its norm. A *skew–product* is the datum of a map f as above and a continuous map $A : U \to B$ defined on a neighborhood of $\mathbb{D}^n_{r_0}$ so that

$$\begin{aligned}(f, A) : \mathbb{T}^n_{r_0} \times B &\to \mathbb{T}^n_{r_0} \times B \\ (x, b) &\mapsto (f(x), A(x)b)\end{aligned}$$

We denote also

$$A^k_x = \prod_{j=1}^{k} A(f^{k-j}(x)) \ ,$$

$$a_k = \int_{\mathbb{T}^n_{r_0}} \log \|A^k_x\| dm(x) \ ,$$

thus the largest Lyapunov exponent is

$$\lambda^{(1)}(f, A) = \liminf_{k \to +\infty} \frac{a_k}{k}.$$

We also recall that the spectral radius of $b \in B$ is given by

$$\rho(b) = \lim_{n \to \infty} \|b^n\|^{1/n} = \inf_{n \geq 1} \|b^n\|^{1/n} \ .$$

Herman's theorem [He3] can be stated as follows :

Theorem 11.9 (Herman's formula) *Under the above assumptions for the skew–product (f, A) one has*

$$\lambda^{(1)}(f, A) \geq \log(\rho(A(0))) \ . \qquad [11.11]$$

Proof. The fundamental remark is that the function $z \mapsto \frac{1}{k} \log \|A^k_z\|$ is plurisubharmonic (see [H1], Section 2.6.1 : indeed one can reduce the problem to the case

$n = 1$ to which the proof given in Section 1.6.6 applies). Thus one has

$$\begin{aligned}\frac{a_k}{k} &= \int_{\mathbb{T}^n_{r_0}} \frac{1}{k} \log \|A_z^k\| dm \geq \frac{1}{k} \log \|A_0^k\| \\ &= \frac{1}{k} \log \|A^k(0)\| \geq \log \rho(A(0)) \, .\end{aligned}$$

□

Appendices

Appendix 1. (by A. Davini) Denjoy theorem and examples

Appendix 2. (by C. Carminati) Kaloshin's proof of Artin–Mazur's theorem

Appendix 3. (by A. Pratelli) The Grobman–Hartman theorem and the stable manifold theorem

Appendix 4. Uniformization, Distorsion and Quasi–conformal maps

In this appendix we recall some elementary and less elementary facts from the theory of conformal and quasi–conformal maps of one complex variable.

A4.1 A nonempty connected open set is called a *region*.

Theorem A4.1 (The Maximum Principle) *If $f(z)$ is analytic and non–constant in a region Ω of the complex plane $\mathbb{C}$, then its absolute value $|f(z)|$ has no maximum in Ω.*

Proof. It is an easy consequence of the fact that non–constant analytic functions map open sets onto open sets. □

The maximum principle implies that if f is defined and continuous on a compact set K and analytic in the interior of K then the maximum of $|f(z)|$ on K is assumed on the boundary of K. Another easy consequence is the following

Exercise A4.2 (Schwarz's Lemma, automorphisms of the disk and of the upper half–plane) *Schwarz's Lemma* : If $|f(z)|$ is analytic for $|z| < 1$ and satisfies the conditions $|f(z)| \leq 1$, $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$ then $f(z) = cz$ with $c \in \mathbb{C}$, $|c| = 1$. *Automorphisms of the disk* : Show that if $f : \mathbb{D} \to \mathbb{D}$ is an automorphism of the disk and $f(0) = 0$ then $|f'(0)| = 1$ and f is a rotation. Deduce from this that the group of automorphisms of the unit disk $\mathbb{D}$ is

$$\mathrm{Aut}\,(\mathbb{D}) = \{z \mapsto T(z) = \frac{az+b}{\overline{b}z+\overline{a}}\,,\ a,b \in \mathbb{C}\,,\ |a|^2 - |b|^2 = 1\}\,.$$

Using the fact that $z \mapsto \frac{z-i}{z+i}$ is a biholomorphism mapping $\mathbb{H}$ onto $\mathbb{D}$ show that the group of automorphisms of $\mathbb{H}$ is $\mathrm{PSL}\,(2,\mathbb{R})$.

A4.2 A mapping f of a region Ω into $\mathbb{C}$ is called *conformal* if it is holomorphic and injective. Such maps are also called *univalent*. Since an analytic map is injective if and only if $f' \neq 0$ if f is univalent in Ω its derivative never vanishes, i.e. it has no critical points inside Ω. The most important result of the theory of conformal maps is certainly the

Theorem A4.3 (Riemann Mapping Theorem) *Given any simply connected region Ω which is not the whole plane, and a point $z_0 \in \Omega$ there exists a unique conformal map (the Riemann map) $f : \mathbb{D} \to \Omega$ such that f is onto, $f(0) = z_0$ and $f'(0) > 0$.*

Exercise A4.4 Drop the requirement $f'(0) > 0$. Then f is not unique but the number $|f'(0)|$ does not depend on f. It is called [Ah1] the *conformal capacity* of Ω with respect to z_0 and it will be denoted $C(\Omega, z_0)$. [Hint : use Exercise A4.2]

One should not think to an arbitrary simply connected region as the "potato" of PDEs but as a rather irregular object : think to $\mathbb{C}\setminus M$ where M is the Mandelbrot set.

A4.3 Assume that the region Ω is bounded and $\partial\Omega$ is a closed Jordan curve (i.e. $\partial\Omega = \gamma([0,1])$, where $\gamma : [0,1] \to \mathbb{C}$ is continuous, $\gamma(0) = \gamma(1)$ and $\gamma(t_1) = \gamma(t_2)$ if and only if $t_1 = t_2$ or $t_1 = 0$, $t_2 = 1$). In this case the Riemann map $f : \mathbb{D} \to \Omega$ has a nice boundary behaviour :

Theorem A4.5 (Caratheodory) *A Riemann map $f : \mathbb{D} \to \Omega$ extends to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{\Omega}$ if and only if $\partial\Omega$ is a closed Jordan curve.*

The topic of the boundary behaviour of conformal maps is very rich and it's an active research area : we refer to [Po] for more informations and references. We will only need two other results : the first extends Caratheodory's theorem dropping the assumption that the restriction of f to the boundary of the disk is injective.

Theorem A4.6 *Let $f : \mathbb{D} \to \Omega$ be a Riemann map. The following four conditions are equivalent :*
(i) f has a continuous extension to $\overline{\mathbb{D}}$;
(ii) $\partial\Omega$ is a continuous curve, i.e. $\partial\Omega = \{\varphi(\zeta)\,, \zeta \in \mathbb{T}\}$ with φ continuous ;
(iii) $\partial\Omega$ is locally connected ;
(iv) $\mathbb{C} \setminus \Omega$ is locally connected.

Our second result, due to Fatou, applies to all $f : \mathbb{D} \to \mathbb{C}$ holomorphic *and bounded* (thus we are dropping the assumption of f being injective in $\mathbb{D}$).

Theorem A4.7 (Fatou) *Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic and bounded. Then f has a non–tangential limit at almost all points $\zeta \in \mathbb{T} = \partial\mathbb{D}$. Moreover if f is not identically zero then $\varphi(\zeta) = \lim_{r\to 1-} f(r\zeta)$ (which belongs to $L^\infty(\mathbb{T})$) is not zero almost everywhere.*

A4.4 Another fundamental result is the celebrated

Theorem A4.8 (Uniformization Theorem) *The only simply connected Riemann surfaces, up to biholomorphic equivalence, are the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the complex plane $\mathbb{C}$ and the unit disk $\mathbb{D}$.*

Exercise A4.9 Prove that the group of automorphisms of the Riemann sphere is the group $\mathrm{PSL}\,(2,\mathbb{C})$ acting by homographies : if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}\,(2,\mathbb{C})$ then $z \mapsto g \cdot z = \frac{az+b}{cz+d}$. The group of automorphisms of the complex plane is simply the affine group.

The plane $\mathbb{C}$, the punctured plane $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ and the two–dimensional torus $\mathbb{T}^2$ are called *parabolic* Riemann surfaces : their universal covering space is conformally

equivalent to the plane. The sphere $\overline{\mathbb{C}}$ is called an *elliptic* Riemann surface. The universal covering spaces of all the other Riemann surfaces are conformally equivalent to $\mathbb{D}$ and these sufaces are called *hyperbolic*. If X is a hyperbolic Riemann surface we will denote d_X the hyperbolic metric on X induced by the Poincaré metric $(ds_{\mathbb{D}})^2 = \frac{4dzd\overline{z}}{(1-|z|^2)^2}$ on the unit disk.

A *branch point* of a holomorphic map f is a critical point of f, i.e. a point z_0 where the derivative of f vanishes. A *ramified point* is a critical value, i.e. the image $w_0 = f(z_0)$ of a branch point z_0, $f'(z_0) = 0$. More precisely, if

$$f(z) = w_0 + a(z - z_0)^k + \dots$$

with $k \geq 1$ and $a \neq 0$ then the integer $k = k(z_0)$ is the *branch index* or *local degree* of f at z_0 and z_0 is a branch point if and only if $k \geq 2$. The *multiplicity* of the branch point is $k - 1$.

A holomorphic map $f : X \to Y$ between Riemann surfaces is called a *covering map* if each point of Y has a connected neighborhood U such that each connected component of $f^{-1}(U) \subset X$ maps onto U by a conformal isomorphism. A map $f : X \to Y$ is *proper* if the inverse image of any compact subset of Y is a compact subset of X. Note that every proper map is finite–to–one and has a well defined finite degree $d \geq 1$. Such a map is also called a *d–fold branched covering*. On the other hand a covering map may well be infinite–to–one.

Exercise A4.10 Show that Scwharz lemma has the following more geometrical formulation : let $f : X \to X$ be a holomorphic endomorphism of a hyperbolic Riemann surface. Then either

- f is a covering map and a local isometry for the hyperbolic metric, and it lifts to an automorphism of the disk, or
- f is a contraction for the hyperbolic metric.

The dynamics of holomorphic endomorphisms of a hyperbolic Riemann surface is quite easy to describe :

Theorem A4.11 *Let X be a hyperbolic Riemann surface and let $f : X \to X$ be a holomorphic endomorphism of X. One of the following 4 possibilities must occur :*

(i) All points of X tend towards a unique attracting fixed point $z_0 \in X$: for all $z \in X$ one has $f^n(z) \to z_0$ as $n \to \infty$ and $|f'(z_0)| < 1$.

(ii) *All points of X tend to infinity : for all $z, w \in X$ one has $d_X(w, f^n(z)) \to \infty$ as $n \to \infty$.*

(iii) *f is conformally conjugate to an irrational rotation and X is the disk $\mathbb{D}$, the punctured disk $\mathbb{D}^*$ or the annulus $\mathbb{A}_r = \{z \in \mathbb{C},\ r < |z| < 1\}$, $0 < r < 1$.*

(iv) *f is a finite order map : there is an $n > 0$ such that $f^n = id$.*

Proof. (sketch : see [Ly2], p. 49 and [McS], section 3, for a complete argument) All hyperbolic riemann surfaces are the quotient $\mathbb{D}/\Gamma$ of the disk by some discrete group Γ of isometries. Lifting f to an analytic endomorphism F of $\mathbb{D}$ and applying the Schwarz Lemma (see also Exercise A4.10) one sees that either F contracts the hyperbolic metric or it is an automorphism of $\mathbb{D}$. In the first case one of the possibilities (i) or (ii) occurs and the two cases correspond to the existence of a fixed point belonging to $\mathbb{D}$ or $\partial\mathbb{D}$ respectively (Wolff–Denjoy point). In the second case if one considers the group Γ_F of automorphisms of $\mathbb{D}$ obtained adjoining F to Γ one can still obtain a discrete group or obtain a non discrete group. If Γ_F is discrete one is again in case (ii) if F has infinite order, whereas if F has finite order one obtains (iv). If Γ_F is not discrete then one can show that Γ (which is discrete) is generated by a single element γ. If γ is hyperbolic one obtains $\mathbb{A}_r$, if it is parabolic $\mathbb{D}^*$ whereas if it is the identity one gets $\mathbb{D}$. Since we have assumed that Γ_F is not discrete and that F is an automorphism f must be an irrational rotation. □

A *branched covering map* between Riemann surfaces is a holomorphic map $f : X \to Y$ such that every point of Y has a connected neighborhood U so that each connected component of $f^{-1}(U)$ maps onto U by a proper map.

Exercise A4.12 (Riemann–Hurwitz formula) Let $f : X \to Y$ be a branched covering map from one compact Riemann surface onto another. Show that the number of branch points, counted with multiplicity, is equal to $\chi(Y)d - \chi(X)$, where χ is the Euler characteristic and d is the degree.

A4.5 One can consider univalent functions f on regions of $\overline{\mathbb{C}}$ with values in $\overline{\mathbb{C}}$: in this case f must be meromorphic and injective. Here are some elementary properties :

Exercise A4.13

(a) If f is univalent on a region $\Omega \subset \overline{\mathbb{C}}$ then f is analytic except for at most a

single simple pole and f' never vanishes.

(b) If $f : \Omega \to \Omega'$ is onto and univalent then $f^{-1} : \Omega' \to \Omega$ is also univalent.

(c) A univalent map is a homeomorphism.

(d) A univalent map preserves angles between curves and their orientation (that's why they're called conformal !).

(e) The composition of univalent maps is univalent ; f is univalent if and only if $1/f$ is univalent.

Lemma A4.14 (Hurwitz) *If a sequence $(f_n)_{n\in\mathbb{N}}$ of functions holomorphic in a region $\Omega \subset \mathbb{C}$ converges uniformly on compact subsets of Ω to a non–constant holomorphic function $f : \Omega \to \mathbb{C}$ then the following statements hold :*

(a) if all the images $f_n(\Omega)$ are contained in a fixed set A then $f(\Omega) \subset A$;

(b) if all the maps $f_n : \Omega \to \mathbb{C}$ are injective then so is $f : \Omega \to \mathbb{C}$;

(c) if all the maps $f_n : \Omega \to \mathbb{C}$ are locally biholomorphic, then so is $f : \Omega \to \mathbb{C}$.

Exercise A4.15 Prove that the space of *all* univalent maps $f : \mathbb{D} \to \overline{\mathbb{C}}$ is compact up to post–composition with automorphisms of $\overline{\mathbb{C}}$. This precisely means that any sequence of univalent maps contains a subsequence $f_n : \mathbb{D} \to \overline{\mathbb{C}}$ such that $M_n \circ f_n$ converges to a univalent map f, uniformly on compact subsets of $\mathbb{D}$, for some sequence of Möbius transforms $M_n \in \mathrm{PGL}\,(2, \overline{\mathbb{C}})$ (see [Mc]).

A4.6 Let $f : \mathbb{C} \to \mathbb{C}$ be a $\mathcal{C}^1$ orientation–preserving diffeomorphism. Then given any point $z_0 \in \mathbb{C}$ one has

$$f(z) = f(z_0) + f_z(z_0)(z - z_0) + f_{\bar z}(z_0)(\bar z - \bar z_0) + \mathrm{o}\,(|z - z_0|)\,, \qquad (A4.1)$$

where

$$f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\,, \quad f_{\bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\,, \quad (z = x + iy)\,. \qquad (A4.2)$$

Note that if f is analytic in z_0 then $f_{\bar z}(z_0) = 0$ (Cauchy–Riemann). The Jacobian determinant of f is $J = |f_z|^2 - |f_{\bar z}|^2$. Since f is orientation–preserving one has $J > 0$, thus $|f_z| > |f_{\bar z}|$.

Definition A4.16 *The* dilatation *of f in z_0 is*

$$D_f(z_0) := \frac{|f_z(z_0)| + |f_{\bar z}(z_0)|}{|f_z(z_0)| - |f_{\bar z}(z_0)|} \geq 1\,. \qquad (A4.3)$$

Note that if f is conformal then $D_f = 1$.

Here is a geometric interpretation of the meaning of the dilatation : the differential $df(z_0)$ maps a circle in the tangent space $T_{z_0}\mathbb{C}$ into an ellipse in $T_{f(z_0)}\mathbb{C}$. The dilatation measures the distorsion since it is the ratio of the major semiaxis amd the minor semiaxis. Indeed applying (A4.1) to an infinitesimal circle $\Delta z = \varepsilon e^{i\theta}$ centered at z_0 one finds an infinitesimal ellipse centered at $f(z_0)$ with major semiaxis $[|f_z| - |f_{\bar{z}}|]^{-1}\varepsilon$ and minor semiaxis $[|f_z| + |f_{\bar{z}}|]^{-1}\varepsilon$.

The *maximal dilatation* of f on $\mathbb{C}$ is $D_f = \sup_{z\in\mathbb{C}} D_f(z)$. If $D_f < +\infty$, let $\kappa_f = (D_f - 1)/(D_f + 1)$. Then one has $\frac{|f_{\bar{z}}|}{|f_z|} \leq \kappa_f < 1$.

Definition A4.17 *f is* quasiconformal *if $D_f < +\infty$, i.e. $\kappa_f < 1$.*

Clearly if f is conformal then $D_f = 1$, $\kappa_f = 0$.

We want now to extend the notion of quasiconformal map to homeomorphisms. We will follow the geometric approach outlined in [Ah2].

Exercise A4.18 Given two rectangles R_1 and R_2 respectively with sides $a_1 \leq b_1$ and $a_2 \leq b_2$, show that there exists a conformal map of R_1 onto R_2 which maps vertices on vertices if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$.

Definition A4.19 *A* quadrilateral *$Q(z_1, z_2, z_3, z_4)$ is a Jordan domain in $\mathbb{C}$ with four distinguished boundary points z_1, z_2, z_3, z_4. Its* modulus *$M(Q)$ is the ratio a/b of the lengths $a < b$ of the sides of any rectangle R which is the conformal image of Q and whose vertices are image of the distinguished points.*

Note that the modulus of a quadrilateral is a conformal invariant. Thus one can use its variation under a homeomorphism to measure the lack of conformality of a map.

Definition A4.20 *Let $f : \mathbb{C} \to \mathbb{C}$ be an orientation–preserving homeomorphism. Its* maximal dilatation *is $D_f := \sup_{Q\subset\mathbb{C},\, Q \text{ quadrilateral}} \frac{M(f(Q))}{M(Q)}$. Let $\kappa_f = (D_f - 1)/(D_f + 1)$. f is a* quasiconformal homeomorphism *if $D_f, +\infty$, i.e. $\kappa_f < 1$.*

Exercise A4.21 Prove that f is conformal if and only if $D_f = 1$.

Theorem A4.22 *A quasiconformal homeomorphism f with maximal dilatation D_f is almost everywhere differentiable and at each point z_0 where f is differentiable*

one has

$$\frac{|f_{\bar{z}}(z_0)|}{|f_z(z_0)|} \leq \kappa_f \ .$$

Perhaps the most useful result in the theory of quasiconformal maps is the following existence theorem also known as Measurable Riemann Mapping Theorem [Ah2, p.98]

Theorem A4.23 *Let μ be a complex–valued measurable function with $\|\mu\|_\infty < 1$. There exists a quasiconformal mapping f such that*

$$f_{\bar{z}} = \mu(z) f_z \text{ almost everywhere} \qquad (A4.4)$$

and f leaves the points $0, 1, \infty$ fixed.

The equation (A4.4) is also known as *Beltrami equation.*

Quasiconformal maps have been introduced in the subject of holomorphic dynamics by Dennis Sullivan and Adrien Douady and have rapidly become a standard tool. What we will need Lecture 6 is the following :

Theorem A4.24 (Douady–Hubbard : stability of the quadratic polynomial) *Let $P_\lambda(z) = \lambda\left(z - \frac{z^2}{2}\right)$ and let $F(z) = P_\lambda(z) + \psi(z)$ where ψ is holomorphic and bounded in the disk $\mathbb{D}_3$, $\psi(z) = \sum_{n=2}^{\infty} \psi_n z^n$ (i.e. $\psi(0) = \psi'(0) = 0$). Assume that $\sup_{z\in\mathbb{D}_3} |\psi(z)| < 10^{-2}$. Then there exists a quasiconformal homeomorphism h such that on the disk $\mathbb{D}_2$ one has $h^{-1} F h = P_\lambda$. If ψ is small enough then h is near the identity in the $\mathcal{C}^0$ topology.*

Appendix 5. Continued fractions, Brjuno numbers

In this appendix we recall some elementary facts on standard real continued fractions (we refer to [MMY], and references therein, for more general continued fractions).

We will consider the iteration of the Gauss map

$$A : (0,1) \mapsto [0,1] \ , \qquad (A5.1)$$

defined by

$$A(x) = \frac{1}{x} - \left[\frac{1}{x} \right] . \tag{A5.2}$$

A is piecewise analytic with branches

$$A(x) = x^{-1} - n \text{ if } \frac{1}{n+1} < x \leq \frac{1}{n} , n \geq 1 .$$

Exercise A5.1 Prove that $A^*(\rho(x)dx) = \rho(x)dx$ where $\rho(x) = [(1+x)\log 2]^{-1}$, i.e. ρ is an invariant probability density for the Gauss map.

Let

$$G = \frac{\sqrt{5}+1}{2} , \; g = G^{-1} = \frac{\sqrt{5}-1}{2} .$$

To each $x \in \mathbb{R} \setminus \mathbb{Q}$ we associate a continued fraction expansion by iterating A as follows. Let

$$\begin{aligned} x_0 &= x - [x] , \\ a_0 &= [x] , \end{aligned} \tag{A5.3}$$

then one obviously has $x = a_0 + x_0$. We now define inductively for all $n \geq 0$

$$\begin{aligned} x_{n+1} &= A(x_n) , \\ a_{n+1} &= \left[\frac{1}{x_n} \right] \geq 1 , \end{aligned} \tag{A5.4}$$

thus

$$x_n^{-1} = a_{n+1} + x_{n+1} . \tag{A5.5}$$

Therefore we have

$$x = a_0 + x_0 = a_0 + \frac{1}{a_1 + x_1} = \ldots = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + x_n}}} , \tag{A5.6}$$

and we will write

$$x = [a_0, a_1, \ldots, a_n, \ldots] . \tag{A5.7}$$

The nth-convergent is defined by

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}} . \tag{A5.8}$$

Exercise A5.2 Show that the numerators p_n and denominators q_n are recursively determined by

$$p_{-1} = q_{-2} = 1 \ , \quad p_{-2} = q_{-1} = 0 \ , \tag{A5.9}$$

and for all $n \geq 0$ one has

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \ , \\ q_n &= a_n q_{n-1} + q_{n-2} \ . \end{aligned} \tag{A5.10}$$

Exercise A5.3 Show that for all $n \geq 0$ one has

$$x = \frac{p_n + p_{n-1} x_n}{q_n + q_{n-1} x_n} \ , \tag{A5.11}$$

$$x_n = -\frac{q_n x - p_n}{q_{n-1} x - p_{n-1}} \ , \tag{A5.12}$$

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n \ . \tag{A5.13}$$

Note that $q_{n+1} > q_n > 0$ and that the sequence of the numerators p_n has the same constant sign of x. Equation (A5.13) implies also that for all $k \geq 0$ and for all $x \in \mathbb{R} \setminus \mathbb{Q}$ one has $\frac{p_{2k}}{q_{2k}} < x < \frac{p_{2k+1}}{q_{2k+1}}$.

Let

$$\beta_n = \Pi_{i=0}^n x_i = (-1)^n (q_n x - p_n) \quad \text{for } n \geq 0, \quad \text{and } \beta_{-1} = 1 \ . \tag{A5.14}$$

Then $x_n = \beta_n \beta_{n-1}^{-1}$ and $\beta_{n-2} = a_n \beta_{n-1} + \beta_n$.

Proposition A5.4 *For all $x \in \mathbb{R} \setminus \mathbb{Q}$ and for all $n \geq 1$ one has*

(i) $|q_n x - p_n| = \dfrac{1}{q_{n+1} + q_n x_{n+1}}$, *so that* $\frac{1}{2} < \beta_n q_{n+1} < 1$;

(ii) $\beta_n \leq g^n$ *and* $q_n \geq \frac{1}{2} G^{n-1}$.

Proof. Using (A5.11) one has

$$\begin{aligned} |q_n x - p_n| &= \left| q_n \frac{p_{n+1} + p_n x_{n+1}}{q_{n+1} + q_n x_{n+1}} - p_n \right| = \frac{|q_n p_{n+1} - p_n q_{n+1}|}{q_{n+1} + q_n x_{n+1}} \\ &= \frac{1}{q_{n+1} + q_n x_{n+1}} \end{aligned}$$

by (A5.13). This proves (i).

Let us now consider $\beta_n = x_0 x_1 \dots x_n$. If $x_k \geq g$ for some $k \in \{0, 1, \dots, n-1\}$, then, letting $m = x_k^{-1} - x_{k+1} \geq 1$,

$$x_k x_{k+1} = 1 - m x_k \leq 1 - x_k \leq 1 - g = g^2 \ .$$

This proves (ii). □

Remark A5.5 Note that from *(ii)* it follows that $\sum_{k=0}^{\infty} \frac{\log q_k}{q_k}$ and $\sum_{k=0}^{\infty} \frac{1}{q_k}$ are always convergent and their sum is uniformly bounded.

For all integers $k \geq 1$, the iteration of the Gauss map k times leads to the following partition of $(0,1)$; $\sqcup_{a_1,\ldots,a_k} I(a_1,\ldots,a_k)$, where $a_i \in \mathbb{N}$, $i = 1,\ldots,k$, and

$$I(a_1,\ldots,a_k) = \begin{cases} \left(\frac{p_k}{q_k}, \frac{p_k+p_{k-1}}{q_k+q_{k-1}}\right) & \text{if } k \text{ is even} \\ \left(\frac{p_k+p_{k-1}}{q_k+q_{k-1}}, \frac{p_k}{q_k}\right) & \text{if } k \text{ is odd} \end{cases}$$

is the branch of A^k determined by the fact that all points $x \in I(a_1,\ldots,a_k)$ have the first $k+1$ partial quotients exactly equal to $\{0, a_1, \ldots, a_k\}$. Thus

$$I(a_1,\ldots,a_k) = \left\{ x \in (0,1) \mid x = \frac{p_k + p_{k-1}y}{q_k + q_{k-1}y}\ ,\ y \in (0,1) \right\}\ .$$

Note that $\frac{dx}{dy} = \frac{(-1)^k}{(q_k+q_{k-1}y)^2}$ is positive (negative) if k is even (odd). It is immediate to check that any rational number $p/q \in (0,1)$, $(p,q) = 1$, is the endpoint of exactly two branches of the iterated Gauss map. Indeed p/q can be written as $p/q = [\bar{a}_1,\ldots,\bar{a}_k]$ with $k \geq 1$ and $\bar{a}_k \geq 2$ in a unique way and it is the left (right) endpoint of $I(\bar{a}_1,\ldots,\bar{a}_k)$ and the right (left) endpoint of $I(\bar{a}_1,\ldots,\bar{a}_k - 1, 1)$ if k is even (odd).

Appendix 6. Rotation number and circle diffeomorphisms

Lemma A6.1 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be monotonic continuous functions such that $f(x+1) = f(x)+1$ and $g(x+1) = g(x)+1$ for all $x \in \mathbb{R}$. Then*

(i) $\rho(f) = \lim_{n\to\infty} \frac{f^n(0)}{n}$ *exists and* $|f^n(0) - n\rho(f)| < 1$;

(ii) $\lim_{n\to\infty} \frac{f^n(x)-x}{n}$ *exists for all* $x \in \mathbb{R}$ *and is equal to* $\rho(f)$;

(iii) $\rho(f) = m/n$ *with* $m, n \in \mathbb{Z}$, $n > 0$ *if and only if there exists* $x \in \mathbb{R}$ *such that* $f^n(x) = x + m$;

(iv) given $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $\|f - g\|_0 = \sup_{x\in\mathbb{R}} |f(x) - g(x)| < \delta$ *then* $|\rho(f) - \rho(g)| < \varepsilon$;

(v) $\rho(f+n) = \rho(f) + n$ *for any integer* n.

Proof. Let $M_k = \sup_{x\in\mathbb{R}}(f^k(x) - x)$ and $m_k = \inf_{x\in\mathbb{R}}(f^k(x) - x)$. We claim that $M_k - m_k < 1$. In fact, as $f(x+1) = f(x)+1$ we have $f^k(x+1) = f^k(x)+1$.

Therefore, $\varphi = f^k - id$ is periodic with period 1. Consequently there exist $x_k, X_k \in \mathbb{R}$ with $0 \leq x_k - X_k < 1$ such that $\varphi(x_k) = m_k$ and $\varphi(X_k) = M_k$. Since f^k is also monotonic nondecreasing we have $f^k(X_k) \leq f^k(x_k)$. Hence $M_k + X_k \leq m_k + x_k$ and so $M_k - m_k \leq x_k - X_k < 1$ which proves our claim.

We are now going to prove that

$$f^k(y) - y - 1 \leq f^k(x) - x \leq f^k(y) - y + 1 \qquad (A6.1)$$

for all $x, y \in \mathbb{R}$. In fact, $f^k(y) - y - 1 \leq M_k - 1 < m_k \leq f^k(x) - x \leq M_k \leq m_k + 1 \leq f^k(y) - y + 1$. We next put $y = 0$ and $x = f^{k(j-1)}(0)$ in (1) and obtain

$$f^k(0) - 1 \leq f^{kj}(0) - f^{k(j-1)}(0) \leq f^k(0) + 1 \; .$$

Thus

$$n(f^k(0) - 1) = \sum_{j=1}^{n} (f^k(0) - 1) \leq \sum_{j=1}^{n} (f^{kj}(0) - f^{k(j-1)}(0)) \leq n(f^k(0) + 1) \; .$$

From this we deduce that

$$nf^k(0) - n \leq f^{kn}(0) \leq nf^k(0) + n \; .$$

We now divide by kn to obtain

$$\frac{f^k(0)}{k} - \frac{1}{k} \leq \frac{f^{kn}(0)}{kn} \leq \frac{f^k(0)}{k} + \frac{1}{k}$$

or

$$\left| \frac{f^{kn}(0)}{kn} - \frac{f^k(0)}{k} \right| \leq \frac{1}{k} \; . \qquad (A6.2)$$

Similarly

$$\left| \frac{f^{kn}(0)}{kn} - \frac{f^n(0)}{n} \right| \leq \frac{1}{n} \; .$$

The sequence $f^k(0)/k$ is a Cauchy sequence because $|f^k(0)/k - f^n(0)/n| \leq 1/k + 1/n$ and so it converges to some limit $\rho(f)$. By letting n tend to ∞ in (2) we see that $|\rho(f) - f^k(0)/k| \leq 1/k$ which proves (i).

By putting $y = 0$ in (1) we obtain $f^k(0) - 1 \leq f^k(x) - x \leq f^k(0) + 1$. Thus, dividing by k and letting k tend to ∞, we see that $(f^k(x) - x)/k$ converges to $\rho(f)$. This proves (ii).

To prove (iii) suppose that there exists $x \in \mathbb{R}$ such that $f^n(x) = x + m$ with $n, m \in \mathbb{Z}$, $n > 0$. Then $f^{kn}(x) = x + km$ and $\rho(f) = m/n$. Conversely let $\rho(f) = m/n$ and suppose that $f^n(x) \neq x + m$ for all $x \in \mathbb{R}$. Then either $f^n(x) - x > m$ for all $x \in \mathbb{R}$ or $f^n(x) - x < m$ for all $x \in \mathbb{R}$. As $f^n - id$ is periodic there exists $a > 0$ such that $f^n(x) - x \geq m + a$ (or $f^n(x) - x \leq m - a$) for all $x \in \mathbb{R}$. Thus $f^{kn}(x) - x \geq km + ka$ (or $f^{kn}(x) - x \leq km - ka$) and so $\rho(f) \geq (m + a)/n$ (or $\rho(f) \leq (m - a)/n$) which is a contradiction.

To prove (iv) we remark that

$$\begin{aligned} |\rho(f) - \rho(g)| &\leq \left|\rho(g) - \frac{g^k(0)}{k}\right| + \left|\frac{g^k(0)}{k} - \frac{f^k(0)}{k}\right| + \left|\frac{f^k(0)}{k} - \rho(f)\right| \\ &\leq \frac{1}{k} + \frac{1}{k}|g^k(0) - f^k(0)| + \frac{1}{k} \, . \end{aligned}$$

Fix an integer k such that $2/k < \varepsilon/2$ and choose $\delta > 0$ such that $|g^k(0) - f^k(0)| < k\varepsilon/2$ if $\|g - f\| < \delta$. It remains to prove that $\rho(f + n) = \rho(f) + n$ if $n \in \mathbb{Z}$: this follows from $(f + n)^k(x) = f^k(x) + kn$ which one can check by induction. □

Let f be a homeomorphism of $\mathbb{R}$ such that $f(x) - x$ is periodic. We give below several equivalent ways, due to Poincaré, to define the rotation number of f.

Theorem A6.2 *The following definitions are equivalent :*

(i) the sequence $\frac{1}{n}(f^n - id)$ converges uniformly on $\mathbb{R}$ to a constant limit $\rho(f)$;

(ii) Let p/q denote a rational number written in irreducible form with $q \geq 1$ and define :

$$D^+(f) = \left\{ \frac{p}{q} \in \mathbb{Q} \, , \, \forall x \in \mathbb{R} \, , \, f^q(x) < x + p \right\} \, ,$$

$$D^-(f) = \left\{ \frac{p}{q} \in \mathbb{Q} \, , \, \forall x \in \mathbb{R} \, , \, f^q(x) > x + p \right\} \, .$$

These two sets form a Dedekind section of $\mathbb{Q}$ defining $\rho(f)$.

(iii) Let μ denote probability measure on the circle invariant under $\bar{f}$. Then

$$\rho(f) = \int_{\mathbb{T}^1} (f(x) - x) d\mu(x) \, .$$

(iv) There exists a real number $\rho(f)$ and a non–decreasing map $h : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$h(x + 1) = h(x) + 1 \, , \quad h(f(x)) = h(x) + \rho(f) \, .$$

Proof. We will only sketch the arguments showing how the various definitions are related. Write $f(x) = x + \phi(x)$. Then for $n \geq 1$ one has

$$f^n(x) - x = \sum_{i=0}^{n-1} \phi \circ \bar{f}^i(x) \; .$$

Let μ be as in (iii) and suppose μ is ergodic. By Birkhoff's ergodic theorem there exists $x_0 \in \mathbb{R}$ such that

$$\lim_{n \to +\infty} \frac{1}{n}(f^n(x_0) - x_0) = \int_{\mathbb{T}^1} (f(x) - x) d\mu(x) \; . \tag{A6.3}$$

On the other hand, as f^n is increasing, one has

$$\max_{y \in \mathbb{R}}(f^n(y) - y) < \min_{y \in \mathbb{R}}(f^n(y) - y) + 1 \; . \tag{A6.4}$$

Putting (1) and (2) together gives the equivalence of (i) and (iii).

In order to prove (iv) take the lift $\tilde{\mu}$ of μ to $\mathbb{R}$ and define

$$h(x) = \begin{cases} \tilde{\mu}([0, x)) & \text{if } x > 0, \\ -\tilde{\mu}([x, 0)) & \text{if } x \leq 0. \end{cases}$$

It is immediate to check that for all $x \in \mathbb{R}$ one has :

$$\begin{aligned} h(x+1) - h(x) &= \tilde{\mu}([x, x+1)) = 1 \; , \\ h(f(x)) - h(x) &= \tilde{\mu}([x, f(x))) = \tilde{\mu}([0, f(0))) = h(f(0)) \; , \\ f^n(0) - 1 < h(f^n(0)) &= nh(f(0)) < f^n(0) + 1 \; . \end{aligned}$$

Thus $h(f(0)) = \rho(f)$ and one obtains (iv).

Finally let $p/q \in D^+(f)$ and $p'/q' \in D^-(f)$. For all $x \in \mathbb{R}$ one has

$$x + p'q < f^{qq'}(x) < x + pq'$$

thus $p/q > p'/q'$. If p/q and p'/q' do not belong to the union $D^+(f) \cup D^-(f)$ there exist $x_0, x_1 \in [0, 1)$ such that $f^q(x_0) = x_0 + p$ and $f^{q'}(x_1) = x_1 + p'$. Then, for $n \geq 1$:

$$-1 + np'q \leq f^{nqq'}(x_1) - 1 < f^{nqq'}(0) \leq f^{nqq'}(x_0) < 1 + npq' \; ,$$

and this gives $p/q \geq p'/q'$. By simmetry one also gets $p'/q' \geq p/q$ hence (ii) holds. □

Exercise A6.3 Prove the following properties of the rotation number :

(i) $\rho(f^n) = n\rho(f)$;

(ii) if f, g are two homeomorphisms of $\mathbb{R}$ such that $f - \mathrm{id}$ and $g - \mathrm{id}$ are $\mathbb{Z}$–periodic and $h : \mathbb{R} \to \mathbb{R}$ is an increasing map which commutes with integer translations and such that $h \circ f = g \circ h$ then $\rho(f) = \rho(g)$: the rotation number is a conjugacy invariant ;

(iii) if $\rho(f)$ is irrational then h of Theorem ? ? (iv) is uniquely determined by its value at zero ;

(iv) under the same assumptions of (iii) prove that the measure μ is unique.

Appendix 7. (by G. Menconi) Universal features of one–parameter families of maps on the interval

Appendix 8. (by R. Stasi) Sinai–Ruelle–Bowen measures

Appendix 9. (by D. Zuddas) Algebraic entropy

References

[AbM] R. Abraham, J. Marsden "Foundations of Mechanics" Benjamin Cummings, New York (1978)

[AG] S. Alinhac, P. Gérard "Opérateurs pseudo–différentiels et thèoréme de Nash–Moser" Savoirs Actuels, CNRS Editions (1991)

[Ah1] L.V. Ahlfors "Conformal Invariants : Topics in Geometric Function Theory" McGraw–Hill (1973)

[Ah2] L.V. Ahlfors "Lectures on Quasiconformal Mappings" Van Nostrand (1966)

[AM] M. Artin and B. Mazur "On periodic points" Ann. of Math. **81** (1965), 82–99

[An] D. V. Anosov "Geodesic flows on closed Riemannian manifolds with negative curvature" Proc. Steklov Inst. Math. **90** (1967), 1–235

[AP] A. Andronov and L. Pontrjagin "Systèmes grossiers" Dokl. Akad. Nauk. SSSR **14** (1937), 247–251

[Ar1] V. I. Arnol'd "On the mappings of the circumference onto itself" Translations of the Amer. Math. Soc. **46**, 2nd series, 213–284 (1961)

[Ar2] V. I. Arnol'd "Small denominators and problems of stability of motion in classical celestial mechanics" Russ. Math. Surv. **18** (1963), 85–193.

[Ar3] V. I. Arnol'd "Instability of dynamical systems with several degrees of freedom" Sov. Math. Dokl. **5** (1964), 581–585.

[Ar4] V. I. Arnol'd "Geometrical Methods in the Theory of Ordinary Differential Equations" Springer–Verlag (1983)

[AKN] V. I. Arnol'd, V. V. Kozlov and A. I. Neishtadt "Dynamical Systems III", Springer–Verlag (1988).

[Ba] I. N. Baker "An entire function which has wandering domains" *J. Austral. Math. Soc.* **22** (1976) 173–176

[Bal] V. Baladi "Dynamical zeta functions" in in "Real and Complex Dynamical Systems" B. Branner and P. Hjorth eds. NATO ASI Series **C464**, Kluwer, Dordrecht (1995), 1–26

[Be] A. Beardon "Iteration of Rational Functions" Springer–Verlag (1991)

[BFGG] G. Benettin, G. Ferrari, L. Galgani and A. Giorgilli "An Extension of the Poincaré–Fermi Theorem on the Nonexistence of Invariant Manifolds in Nearly Integrable Hamiltonian Systems" Il Nuovo Cimento **72B** (1982) 137

[BG] G. Benfatto and G. Gallavotti "Renormalization Group" Princeton University Press (1995)

[BHS] H. W. Broer, G. B. Huitema and M. B. Sevryuk "Quasi–Periodic Motions in Families of Dynamical Systems" Springer–Verlag (1996)

[BKS] T. Bedford, M. Keane and C. Series (Editors) "Ergodic theory, symbolic dynamics, and hyperbolic spaces" Papers from the Workshop on Hyperbolic Geometry and Ergodic Theory held in Trieste, April 17–28, 1989. Oxford Science Publications (1991)

[BM] A. Berretti and S. Marmi "Scaling, Perturbative Renormalization and Analiticity for the Standard Map and some Generalizations" *Chaos, Solitons and Fractals* **5** (1995) 257–269

[Bs] J. B. Bost "Tores invariants des systèmes dynamiques hamiltoniens" Séminaire Bourbaki **639**, Astérisque **133–134** (1986), 113–157.

[Br] A. D. Brjuno "Analytical form of differential equations" *Trans. Moscow Math. Soc.* **25** (1971), 131-288 ; **26** (1972), 199-239.

[Bo] R. Bowen "On Axiom A Diffeomorphisms" *Regional Conference Series in Mathematics* **35**, AMS, Providence, RI, (1978)

[CG] L. Carleson and T. Gamelin "Complex Dynamics" Universitext, Springer–Verlag, Berlin Heidelber New York (1993)

[CGSY] D. Cervau, E. Ghys, N. Sibony and J.–C. Yoccoz "Dynamique et Géométrie Complexes" Panoramas et Synthèses **8** Soc. Math. France (1999)

[Ch] T.M. Cherry "Analytic quasiperiodic curves of discontinuous type on a torus" *Proc. London Math. Soc. (2nd series)* **44** (1938), 175–215

[CL] T. Carletti and J. Laskar "Scaling Law in the standard map critical function. Interpolating Hamiltonian and frequency map analysis" Nonlinearity **13** 2033–2061 (2000)

[CM] T. Carletti and S. Marmi "Linearization of Analytic and Non–Analytic Germs of Diffeomorphisms of $(\mathbb{C}, 0)$" Bulletin de la Societé Mathématique de France (1999)

[Cv] P. Cvitanović "Universality in Chaos" Adam Hilger Ltd (1984)

[Da] A.M. Davie "The critical function for the semistandard map" *Nonlinearity* **7** (1994), 219 - 229.

[De] A. Denjoy "Sur les courbes définies par les équations différentielles à la surface du tore" *J. Math. Pures et Appl.* **11** (1932), 333-375

[DJ] C. Di Castro and G. Jona Lasinio "On the microscopic foundations of the scaling laws" *Phys. Lett.* **29A** (1969) 322–323

[DM] W. De Melo "Rigidity and Renormalization in One Dimensional Dynamical Systems" *Doc. Math. J. DMV* Extra Volume ICM 1998 II 765–778

[Do] A. Douady "Disques de Siegel et anneaux de Herman" Séminaire Bourbaki n. 677, *Astérisque* **152–153** (1987), 151–172

[Du] R. Duarte "Plenty of elliptic islands for the Standard family" Ann. Inst. Henri Poincaré Analys. Non Lineaire **11** 359–409 (1994)

[Fal] K. Falconer "Fractal Geometry. Mathematical Foundations and Applications" John Wiley and Sons (1990)

[Fa1] A. Fathi "Thórème KAM faible et théorie de Mather sur les systèmes lagrangiens" C.R. Acad. Sci. Paris Sér. I Math. **324** 1043–1046 (1997)

[Fa2] A. Fathi "Solutions KAM faibles conjuguées et barrières de Peirels" C.R. Acad. Sci. Paris Sér. I Math. **325** 649–652 (1997)

[Fa3] A. Fathi "Orbites hétéroclines et ensemble de Peirels" C.R. Acad. Sci. Paris Sér. I Math. **326** 1213–1216 (1998)

[Fa4] A. Fathi "Sur la convergence du semi–groupe de Lax–Oleinik" C.R. Acad. Sci. Paris Sér. I Math. **327** 267–270 (1998)

[FaM] A. Fathi and J.N. Mather "Failure of convergence of the Lax–Oleinik semi-group in the time–periodic case" preprint (2000)

[F] P. Fatou "Sur les équations fonctionnelles" *Bull. Soc. Math. France* **47** (1919) 161–271 ; **48** (1920) 33–94 and 208–314

[Fay] B. R. Fayad "Analytic mixing reparametrizations of irrational flows on the torus $\mathbb{T}^n$, $n \geq 3$" Ergodic Th. and Dyn. Sys. (to appear, see also "Reparamétrage de flots irrationels sur le tore" Thèse de doctorat, Ecole Polytechnique (2000))

[Ga1] G. Gallavotti "Quasi–Integrable Mechanical Systems" in Phénomènes critiques, systèmes aléatoires, théories de jauge, Part I, II, Les Houches 1984, North–Holland, Amsterdam (1986) 539–624

[Ga2] G. Gallavotti "Quasi Periodic Motions From Hipparcus to Kolmogorov" preprint IHES/P/99/74 (1999)

[Ge] V.G. Gelfreich "A proof of the exponentially small transversality of the separatrices for the Standard map" Commun. Math. Phys. **201** 155–216 (1999)

[GM] A. Giorgilli and A. Morbidelli "Invariant KAM tori and global stability for Hamiltonian systems" ZAMP **48** (1997), 102–134.

[GJ] J. Glimm and A. Jaffe "Quantum Physics. A Functional Integral Point of View" second edition, Springer–Verlag (1987)

[Gr] M.L. Gromov "Smoothing and Inversion of Differential Operators" *Math. USSR Sbornik* **17** (1972), 381–434

[Ha] R.S. Hamilton "The Inverse Function Theorem of Nash and Moser" *Bull. A.M.S.* **7** (1982), 65–222.

[Hay] S. Hayashi "Connecting invariant manifolds and the solution of the $\mathcal{C}^1$ stability and Ω stability conjectures for flows" *Ann. of Math.* **145** (1997), 81–137

[H1] L. Hörmander "Complex analysis in several variables" D. Van Nostrand, New York, (1966)

[H2] L. Hörmander "The Analysis of Linear Partial Differential Operators I" Grundlehren der mathematischen Wissenschaften **256**, Springer–Verlag, Berlin, Heidelberg, New York, Tokyo (1983)

[He1] M. R. Herman "Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations" *Publ. Math. IHES* **49** (1979) 5–233

[He2] M. R. HERMAN - *Sur les courbes invariantes par les difféomorphismes de l'anneau*, Astérisque **103–104** (1983).

[He3] M. R. Herman "Une méthode pour minorer les exposants de Lyapounov et quelques examples montrant le charactère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2" Comment. Math. Helv. **58** 453–502 (1983)

[He4] M. R. Herman "Examples de fractions rationelles ayant une orbite dense sur la sphere de Riemann" Bulletin de la Societé Mathématique de France **112** (1984), 93–142

[He5] M. R. Herman "Simple proofs of local conjugacy theorems for diffeomorphisms of the circle with almost every rotation numbers" Bull. Soc. Bras. Mat. **16** (1985) 45–83

[He6] M. R. Herman "Are there critical points one the boundary of singular domains ?" Commun. Math. Phys. **99** (1985) 593–612.

[He7] M. R. Herman "Recent results and some open questions on Siegel's linearization theorem of germs of complex analytic diffeomorphisms of $\mathbf{C}^n$ near a fixed point" Proc. VIII Int. Conf. Math. Phys. Mebkhout and Seneor eds. (Singapore : World Scientific) (1986), 138–184.

[He8] M. R. Herman "Démonstration du théorème des courbes translatées par difféomorphismes de l'anneau ; démonstration du théorème des tores invariants" manuscripts (1980) and "Abstract methods in small divisors : implicit function theorems in Fréchet spaces", lectures given at the CIME conference on Dynamical Systems and Small Divisors, Cetraro 1998

[Hu] J. H. Hubbard "Local connectivity of Julia sets and bifurcation loci : three theorems of J.–C. Yoccoz" In L. R. Goldberg and A. V. Phillips, editors, *Topological Methods in Modern Mathematics* pp. 467–511 Publish or Perish

Inc (1993)

[L1989] J. LASKAR - *A numerical experiment on the chaotic behaviour of the Solar System*, Nature **338** (1989), 237–238.

[HL] G. H. Hardy and J. E. Littlewood "Notes on the theory of series (XXIV) : a curious power series" Proc. Cambridge Phil. Soc. **42** (1946), 85–90

[HW] G.H. Hardy and E.M. Wright "An introduction to the theory of numbers" Fifth Edition, Oxford Science Publications (1990)

[ID] C. Itzykson and J. M. Drouffe "Statistical Field Theory. Volume 1 From Brownian Motion to Renormalization and Lattice Gauge Theory" Cambridge Monographs on Mathematical Physics (1989)

[Ir] M. C. Irwin "Smooth Dynamical Systems" Academic Press (1980)

[K1] A. N. Kolmogorov "On the persistence of conditionally periodic motions under a small perturbation of the Hamilton function" Dokl. Akad. Nauk SSSR **98** (1954) 527–530 (in Russian : English translation in G. Casati and J. Ford, editors, Stochastic Behavior in Classical and Quantum Hamiltonian Systems, Lecture Notes in Physics **93** (1979) 51–56 Springer–Verlag).

[K2] A. N. Kolmogorov, "The General Theory of Dynamical Systems and Classical Mechanics", address to the 1954 International Congress of Mathematicians, Amsterdam.

[Kad] L. P. Kadanoff "Scaling Laws for Ising Models near T_c" *Physics* **2** (1966) 263–272

[Kal1] V. Yu. Kaloshin "An extension of the Artin–Mazur theorem" Ann. of Math. **150** (1999), 729–741

[Kal2] V. Yu. Kaloshin "Generic Diffeomorphisms with Superexponential Growth of Number of Periodic Orbits" Stony Brook IMS Preprint 1999/2, avaliable on the web (math.DS/9902159, front.math.ucdavis.edu)

[Ka] T. Kato "Perturbation Theory for Linear Operators" Grundlehren der mathematischen Wissenschaften **132**, Springer–Verlag (1984)

[Kat] Y. Katznelson "An Introduction To Harmonic Analysis" Dover Publ. Inc., New York (1976)

[KH] A. Katok and B. Hasselblatt "Introduction to the modern theory of dynamical systems" Encyclopedia of Mathematics and its Applications **54**, Cambridge University Press, (1995)

[Ko] A. V. Kočergin "The absence of mixing in special flows over a rotation of the circle and in flows on a two–dimensional torus" Dokl. Akad. Nauk SSSR **205** (1972) 515–518

[La] S. Lattès "Sur l'iteration des substitutions rationelles et les fonctions de Poincaré" *C. R. Acad. Sci. Paris* **166** (1918) 26–28

[Lo] P. Lochak "Canonical perturbation theory via simultaneous approximations", Russ. Math. Surv. **47** (1992), 57–133.

[Ly1] M. Yu. Lyubich "Entropy of analytic endomorphisms of the Riemannian sphere" *Funct. Anal. and its Appl.* **15** 300–302 (1981)

[Ly2] M. Yu. Lyubich "The dynamics of rational transforms : the topological picture" Russ. Math. Surv. **41** (1986), 43–117

[Ma1] J. N. Mather "Anosov diffeomorphisms" Bull. Amer. Math. Soc. **73** (1967), 792–795

[Ma2] J. N. Mather "Existence of quasi–periodic orbits for twist homeomorphisms of the annulus" Topology **21** 457–467 (1982)

[MaF] J.N. Mather and G. Forni "Action Minimizing Orbits in Hamiltonian Systems" Lecture Notes in Mathematics **1589** 92–186 (1994)

[Mar1] S. Marmi "Chaotic Behaviour in the Solar System (Following J. Laskar)" Séminaire Bourbaki n. 854, Astérisque **266** (2000) 113–136

[Mar2] S. Marmi "An introduction to small divisor problems" Quaderni del Dottorato di Ricerca in Matematica, Pisa (2000) (also available at mp-arc and front.math.ucdavis.edu)

[Man1] A. Manning "Axiom A diffeomorphisms have rational zeta functions" *Bull. London Math. Soc.* **3** (1971) 215–220

[Man2] A. Manning "There are no new Anosov diffeomorphisms on tori" Amer. J. Math. **96** (1974), 422–429

[MM] L. Markus and K. R. Meyer "Generic Hamiltonian Systems are neither integrable nor ergodic" Memoirs of the A.M.S. **144** (1974)

[MMY] S. Marmi, P. Moussa and J.–C. Yoccoz "The Brjuno functions and their regularity properties" *Commun. Math. Phys.* **186** (1997), 265-293

[MMY2] S. Marmi, P. Moussa and J.–C. Yoccoz "Complex Brjuno Functions" Journal of the A.M.S. **14** (2001) 783–841

[Mc1] C. T. McMullen "Complex Dynamics and Renormalization" Ann. of Math. Studies **135**, Princeton University Press (1994)

[Mc2] C. T. McMullen "Frontiers in Complex Dynamics" *Bull. A.M.S.* **31** 155–172 (1994)

[Mc3] C.T. McMullen "Renormalization and 3–manifolds which fiber over the circle" Ann. of Math. Studies **142** Princeton University Press (1996)

[Mc4] C.T. McMullen "Rigidity and Inflexibility in Conformal Dynamics" *Doc. Math. J. DMV* Extra Volume ICM 1998 II 841–855

[McS] C.T. McMullen and D. Sullivan "Quasiconformal homeomorphisms and dynamics III : The Teichmüller space of a holomorphic dynamical system" *Adv. Math.* **135** (1998) 351–395

[Mn1] R. Mañe "Ergodic Theory and Differentiable Dynamics" Springer–Verlag (1987)

[Mn2] R. Mañe "A proof of the $\mathcal{C}^1$ stability conjecture" *Publ. Math. I.H.E.S.* **66** 161–210 (1988)

[Mn3] R. Mañe "On the instability of Herman rings" *Invent. Math.* **81** (1985) 459–471

[M] J. Moser "A rapidly convergent iteration method and nonlinear differential equations" *Ann. Scuola Norm. Sup. Pisa* **20** (1966) 499-535

[Mi] J. Milnor "Dynamics in One Complex Variable : Introductory Lectures" Inst. Math. Sci. SUNY Stony Brook (1990)

[MP] M. Misiurewicz and F. Przytycki "Topological entropy and degree of smooth mappings" *Bull. Acad. Polon. Sci., Ser. Math. Astron. Phys.* **25** 573–574 (1977)

[MSS] R. Mañe, P. Sad and D. Sullivan "On the dynamics of rational maps" *Ann. Sci. École Norm. Sup.* **16** 193–217 (1983)

[N1] J. F. Nash Jr. "Real Algebraic Manifolds" *Ann. of Math.* **56** (1952) 405–421

[N2] J. Nash "The embedding problem for Riemannian manifolds" *Ann. of Math.* **63** (1956) 20–63

[Nek] N.N. Nekhoroshev "An exponential estimate for the time of stability of nearly integrable Hamiltonian systems" *Russ. Math. Surveys* **32** (1977), 1–65.

[Ne1] S. Newhouse "Non–density of Axiom A (a) on $\mathbb{S}^2$" Global Analysis, Proc. Symp. in Pure Math. **14** (1970) 191–202

[Ne2] S. Newhouse "Diffeomorphisms with infinitely many sinks" *Topology* **13** (1974), 9–18

[Ne3] S. Newhouse "The abundance of wild hyperbolic sets and non smooth stable sets of diffeomorphisms" *Publ. Math. I.H.E.S.* **50** (1979), 101–151

[Ne4] S. Newhouse "Lectures on Dynamical Systems" Progr. in Math. **8** (Birkhauser, Boston) 1–114

[OW] D.S. Ornstein and B. Weiss "Statistical Properties of Chaotic Systems" *Bull. A.M.S.* **24** (1991), 11–116

[Ox] Oxtoby "Measure and Category" Graduate Texts in Math. **2** Springer Verlag (1980)

[P] H. Poincaré "Les Méthodes Nouvelles de la Mécanique Celeste", tomes I–III Paris Gauthier–Villars (1892, 1893, 1899).

[Pa] J. Palis "A Global View of Dynamics and a Conjecture on the Denseness of Finitude of Attractors" *Astérisque* **261** (2000), 335–347

[PDM] J. Palis and W. de Melo "Geometric Theory of Dynamical Systems. An Introduction" Springer–Verlag (1982)

[Pe] M. M. Peixoto "Structural stability on two–dimensional manifolds" Topology **2** (1962), 101–121

[PM] R. Pérez–Marco "Solution complète au problème de Siegel de linéarisation d'une application holomorphe au voisinage d'un point fixe (d'après J.–C. Yoccoz)" Séminaire Bourbaki n. 753, *Astérisque* **206** (1992), 273–310

[Po] Ch. Pommerenke "Boundary Behaviour of Conformal Maps" Grundlehren der Mathematischent Wissenschaften **299**, Springer–Verlag (1992)

[Pö] J. Pöschel "Integrability of Hamiltonian systems on Cantor sets" Comm. Pure Appl. Math. **35** 653–696 (1982)

[PT] J. Palis and F. Takens "Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations" Cambridge University Press (1993)

[PY] J. Palis and J.–C. Yoccoz "Differentiable conjugacy of Morse Smale diffeomorphisms" Bol. Soc. Brasil. Mat. **25** 25–48 (1990)

[Re] R. Remmert "Classical Topics in Complex Function Theory" Graduate Texts in Mathematics **172**, Springer–Verlag (1998)

[Ro] J. Robbin "A structural stability theorem" *Ann. of Math.* **94** (1971), 447–493

[Rob] C. Robinson "Structural stability of vector fields" *Ann. of Math.* **99** (1974), 154–175

[Rü] H. Rüssmann "Kleine Nenner II : Bemerkungen zur Newtonschen Methode" Nachr. Akad. Wiss. Göttingen Math. Phys. Kl (1972) 1–20

[S] C.L. Siegel "Iteration of analytic functions" Annals of Mathematics **43** (1942), 807-812.

[Sch] A.J. Schwartz "A generalization of Poincaré–Bendixon theorem to closed two–dimensional manifolds" *Amer. J. of Math.* **85** (1963), 453–458

[Ser] F. Sergeraert "Un théorème de fonctions implicites sur certains espaces de Fréchet et quelques applications" *Ann. Scient. Èc. Norm. Sup.* **5** (1972), 599–660.

[Si] Ya.G. Sinai "Topics in Ergodic Theory" Princeton Mathematical Series **44** (1994)

[Sm1] S. Smale "Morse inequalities for a dynamical system" Bull. Amer. Math. Soc. **66** (1960) 43–49

[Sm2] S. Smale "Structurally stable systems are not dense" Amer. J. Math. **88** (1966), 491–496

[Sm3] S. Smale "Differentiable Dynamical Systems" *Bull. Amer. Math. Soc.* **73** (1967), 747–817

[So] J. Sotomayor "Lições de equações diferenciais ordinárias" Projeto Euclides, IMPA (1979)

[SZ] D. Salomon and E. Zehnder "KAM theory in confuguration space" *Comm. Math. Helvetici* **64**, (1989), 84–132

[St] S. Sternberg "Celestial Mechanics" (two volumes) W.A. Benjamin, New York (1969).

[Su1] D. Sullivan "On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions" in "Riemann Surfaces and Related Topics" Ann. Math. Studies **97** Princeton University Press (1981)

[Su2] D. Sullivan "Quasiconformal homeomorphisms and dynamics I : Solution of the Fatou–Julia problem on wandering domains" *Ann. of Math.* **122** 401–418 (1985)

[Su3] D. Sullivan "Quasiconformal homeomorphisms and dynamics II : Structural stability implies hyperbolicity for Kleinian groups" *Acta Math.* **155** (1985) 243–260

[Su4] D. Sullivan "Bounds, Quadratic Differentials and Renormalization Conjectures" in F. Browder, editor, *Mathematics into the Twenty–first Century : 1988 Centennial Symposium, August 8–12*, pp. 417–466, Amer. Math. Soc. (1992)

[Vi] M. Viana "Dynamics : a probabilistic and geometric perspective" Doc. Math. J. DMV – Extra Volume ICM 1998 I 557–578

[W] K. G. Wilson "The renormalization group and critical phenomena" Nobel Prize Lecture, *Reviews of Modern Physics* **55** (1983) 583–600

[Wa] P. Walters "An Introduction to Ergodic Theory" Graduate Texts in Mathematics **79**, Springer–Verlag (1982)

[WW] E.T. Whittaker and G.N. Watson "A course of modern analysis" Cambridge University Press (1962)

[Wi] A. Wightman "Some Lessons of Renormalization Theory" in *The Lesson of*

Quantum Theory, J. de Boer, E. Dal and O. Ulfbeck editors, Elsevier Science Publishers (1986), 201–226

[You] L.S. Young "Ergodic Theory of Differentiable Dynamical Systems" in "Real and Complex Dynamical Systems" B. Branner and P. Hjorth eds. NATO ASI Series **C464**, Kluwer, Dordrecht (1995), 293–336.

[Yo1] J.–C. Yoccoz "Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne" *Ann. Sc. E.N.S.* **17** (1984) 333–359

[Yo2] J.–C. Yoccoz "An introduction to small divisors problems" in "From number theory to physics", M. Waldschmidt, P. Moussa, J.M. Luck and C. Itzykson (editors) Springer–Verlag (1992), 659–679

[Yo3] J.–C. Yoccoz "Théorème de Siegel, nombres de Bruno et polynômes quadratiques" *Astérisque* **231** (1995), 3-88.

[Yo4] J.–C. YOCCOZ - *Introduction to Hyperbolic Dynamics* in "Real and Complex Dynamical Systems" B. Branner and P. Hjorth eds. NATO ASI Series **C464**, Kluwer, Dordrecht (1995), 265–291

[Yo5] J.–C. Yoccoz "Recent Developments in Dynamics" Proceedings of the ICM Zurich 1994, Birkhäuser Verlag (1995), 246–265

[Yo6] J.–C. Yoccoz, lectures given at the CIME conference on Dynamical Systems and Small Divisors, Cetraro 1998, to appear in Lecture Notes in Mathematics

[Ze1] E. Zehnder "Generalized Implicit Function Theorems with Applications to some Small Divisor Problems (I and II)" *Commun. Pure Appl. Math.* **28** (1975) 91–140, **29** (1976) 49–113.

[Ze2] E. Zehnder "A simple proof of a generalization of a Theorem by C. L. Siegel" Lecture Notes in Mathematics **597** (1977) 855–866.

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List of symbols

Ad : adjoint action
$c(\Omega, z_0)$: conformal capacity of Ω w.r.t. z_0
$\mathbb{C}$: complex plane
$\mathbb{C}^*$: $\mathbb{C} \setminus \{0\}$
$\overline{\mathbb{C}}$: Riemann sphere
$\mathbb{C}\{z\}$: ring of convergent power series in one complex variable
$\mathbb{C}[[z]]$: ring of formal power series in one complex variable
Cent : centralizer
$\widehat{\text{Cent}}$: formal centralizer
$D_f(z_0)$, D_f : dilatiation of f (maximal)
$\mathbb{D}$: open unit disk $\{|z| < 1\}$
$\mathbb{D}_r$: open disk $\{|z| < r\}$ of radius $r > 0$.
$\mathbb{E}$: outer disk $\{|z| > 1\}$
$[f]$: orbit of f
$F(R)$: Fatou set of R
G : group of germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$
$\hat{G}$: formal analogue of G
G_λ : elements of G with linear part λ
$\hat{G}_\lambda$: formal analogue of G_λ
$J(R)$: Julia set of R
$\mathbb{N}$: non–negative integers
Ω : a region of $\mathbb{C}$
$\mathbb{Q}$: rational integers
R_λ : the germ $R_\lambda(z) = \lambda z$
S : univalent maps on $\mathbb{D}$
S_λ : elements of S with linear part R_λ
$S_{\mathbb{S}^1}$: elements of S with linear part of unit modulus
$\mathbb{S}^1$: unit circle $\{|z| = 1\}$
u Yoccoz's function, see Section 3.1
$\mathcal{Y}$: see Chapter 4
$\mathbb{Z}$: integers

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